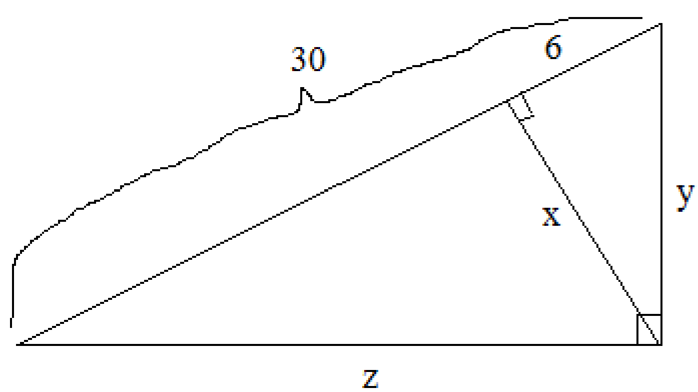
Geometric Mean and Proportional Right Triangles

Notes, Examples, and Practice Exercises (with Solutions)



Topics include geometric mean, similar triangles, Pythagorean Theorem, 45-45-90, 30-60-90, and more.

Cross Product & Similar Right Triangles

Using Cross Products to compare fractions

If $\frac{A}{B} = \frac{C}{D}$ then $AD = BC$

Example: $\frac{3}{4} = \frac{12}{16}$ ---> $3 \times 16 = 4 \times 12 = 48$

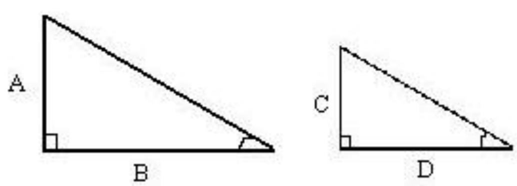
$\frac{A}{B} = \frac{C}{D}$ multiply both sides by B $A = \frac{BC}{D}$ multiply both sides by D $AD = BC$

If $\frac{A}{B} = \frac{C}{D}$ then $\frac{A}{C} = \frac{B}{D}$

Example: $\frac{5}{9} = \frac{25}{45}$ ---> $\frac{5}{25} = \frac{9}{45}$

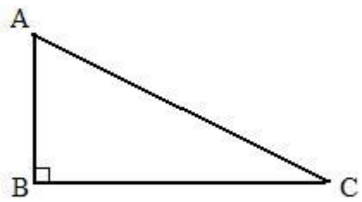
$\frac{A}{B} = \frac{C}{D}$ multiply both sides by B $A = \frac{BC}{D}$ divide both sides by C $\frac{A}{C} = \frac{B}{D}$

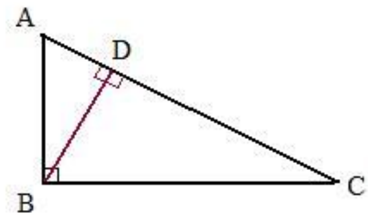
Application: similar right triangles



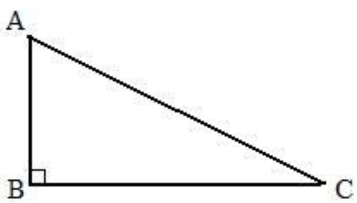
For these similar triangles, the above ratios apply!

Notes on Means-Extremes, Proportions, & Right Triangles

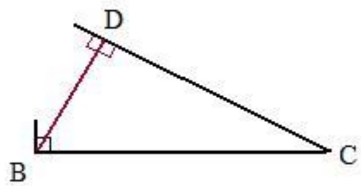


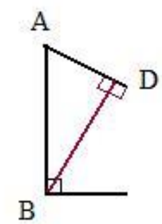


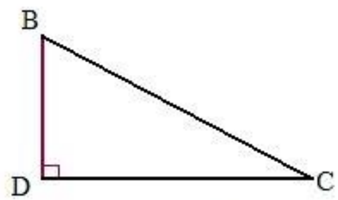
Draw an altitude to hypoteneuse.
Three similar right triangles are formed.



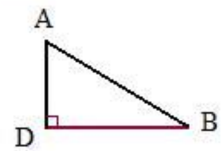
Large Right Triangle







Medium Right Triangle

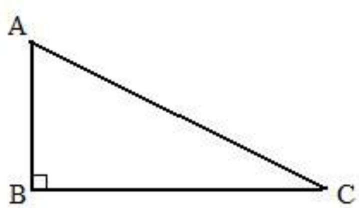


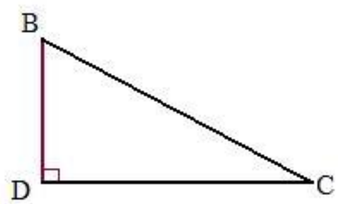
Small Right Triangle

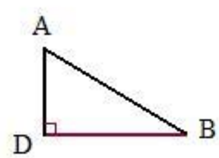
$\triangle ABC \sim \triangle BDC \sim \triangle ADB$

3 similar triangles: each pair can be proven using (AA) Angle-Angle -- Triangle Similarity Theorems

Since the right triangles are similar, the ratios of their sides are the same.







There are numerous ratios that can be written.

$$\frac{\text{left leg}}{\text{hypoteneuse}} = \frac{AB}{AC} = \frac{BD}{BC} = \frac{AD}{DB}$$

$$\frac{AB}{AD} = \frac{AC}{AB} \longrightarrow AB^2 = AC \cdot AD$$

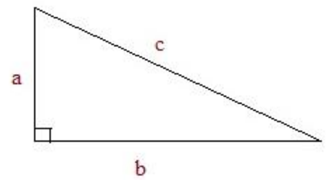
Examples include:

$$\frac{\text{left leg (big)}}{\text{left leg (med)}} \quad \frac{AB}{BD} = \frac{AC}{BC} \quad \frac{\text{hypo (big)}}{\text{hypo (med)}}$$

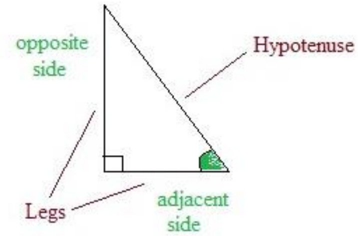
(note: using triangle similarity ratios, one can derive the pythagorean theorem)

Special Right Triangles

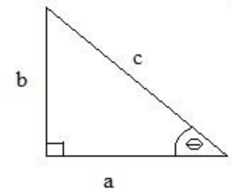
Review Notes:



Pythagorean Theorem: $a^2 + b^2 = c^2$



Trigonometry Relations:



$\sin\theta = \frac{b}{c}$ $\quad \csc\theta = \frac{c}{b}$

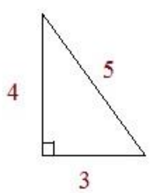
$\cos\theta = \frac{a}{c}$ $\quad \sec\theta = \frac{c}{a}$

$\tan\theta = \frac{b}{a}$ $\quad \cot\theta = \frac{a}{b}$

> Utilizing the Pythagorean Theorem or Trig Identities can find angle and side measurements.
> However,
> "Special Right Triangles" have features that made calculations easy!!

Special Right Triangles:

"Sides"



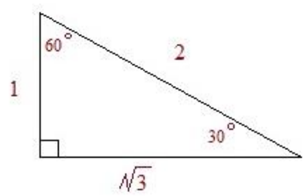
3 - 4 - 5
Right Triangle

Others include: 5 - 12 -13
7 - 24 - 25
8 - 15 -17

> Note:
> -- Pythagorean theorem confirms
> $3^2 + 4^2 = 5^2$
> -- <u>Any multiple of 3-4-5</u> will work!
> Examples: 30-40-50 or 15-20-25

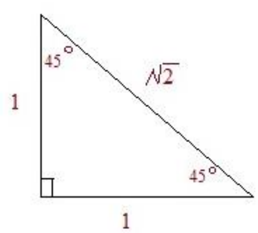
"Angles"

30 - 60 - 90
Right Triangle



> Note:
> -- Pythagorean theorem and trig relations confirm
> (ex: $\sin 30^\circ = 1/2 = .5$)
> -- any ratio of $1 - \sqrt{3} - 2$ will work.
> $\longrightarrow x - \sqrt{3}\,x - 2x$

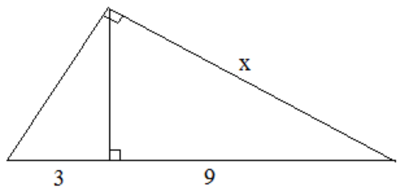
45 - 45 - 90
Right Triangle



> Note:
> -- Pythagorean theorem and trig relations confirm
> -- Congruent sides imply congruent (opposite) angles
> -- any ratio of $1 - 1 - \sqrt{2}$ will work.
> $\longrightarrow x - x - \sqrt{2}\,x$

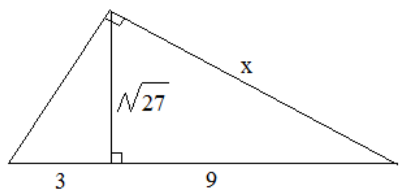
Right Triangles: Altitude, Geometric Mean, and Pythagorean Theorem

Example: Find x:



Step 1: Find the length of the altitude...

$\frac{3}{h} = \frac{h}{9}$ $h = \sqrt{27}$



Geometric mean of divided hypotenuse is the length of the altitude

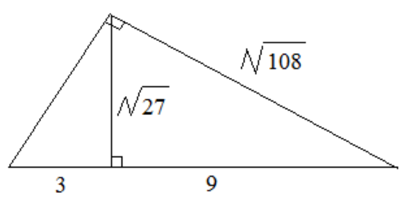
$\sqrt{27}$ is the geometric mean of 3 and 9

Step 2: Find x

$\sqrt{27}^2 + 9^2 = x^2$

$27 + 81 = x^2$

$x = \sqrt{108}$



Pythagorean Theorem:

$a^2 + b^2 = c^2$ *where a and b are legs and c is the hypotenuse.*

Step 3: Check solution (with other sides)

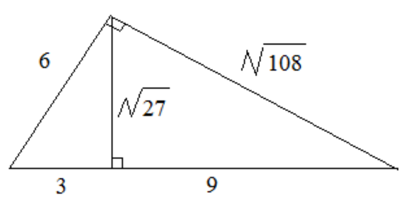
$3^2 + \sqrt{27}^2 = c^2$

$c = 6$

Then,

$6^2 + \sqrt{108}^2 = 12^2$

$36 + 108 = 144$ ✓



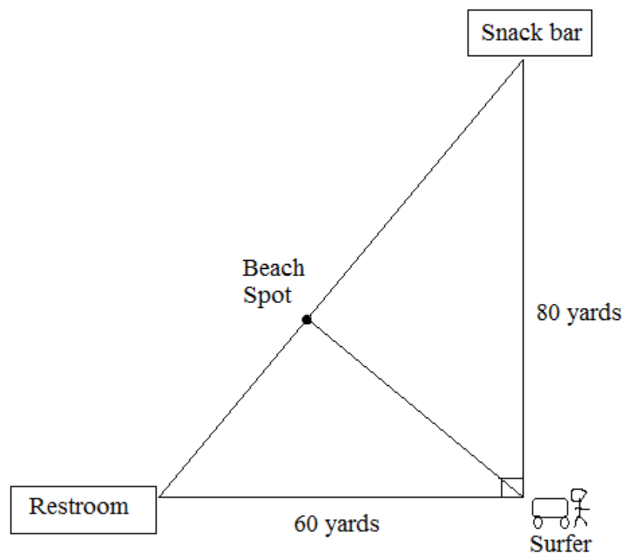
(all 3 right triangles satisfy the Pythagorean Theorem)

Altitude to Hypotenuse, Proportions, and Pythagorean Theorem

A surfer wants to walk directly to the beach from his car. (see diagram)

a) What is the shortest distance to the beach?

b) How far is the beach spot from the snack bar?



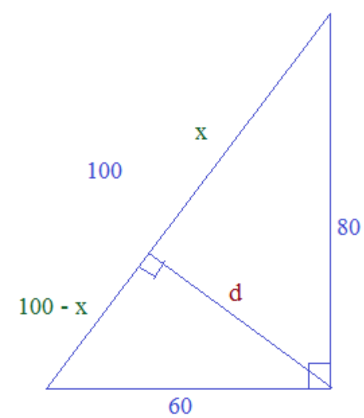
*** The walk directly to the beach will form a right angle (i.e. creating altitude to hypotenuse)

*** The distance from Restroom to Snack Bar is 100 yds. (Pythagorean Theorem)

a) Recognizing "altitude to hypotenuse" cuts right triangle into 3 similar right triangles....

	medium triangle		large triangle
$\frac{\text{hypotenuse}}{\text{small leg}}$	$\frac{80}{d}$	=	$\frac{100}{60}$

$d = 48$



b) Then, to find distance from beach spot to snack bar (x) we know that d is the geometric mean between x and 100 - x...

$$\frac{100 - x}{d} = \frac{d}{x}$$

$$\frac{100 - x}{48} = \frac{48}{x}$$

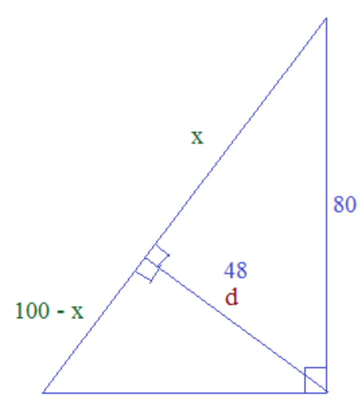
$$2304 = 100x - x^2$$

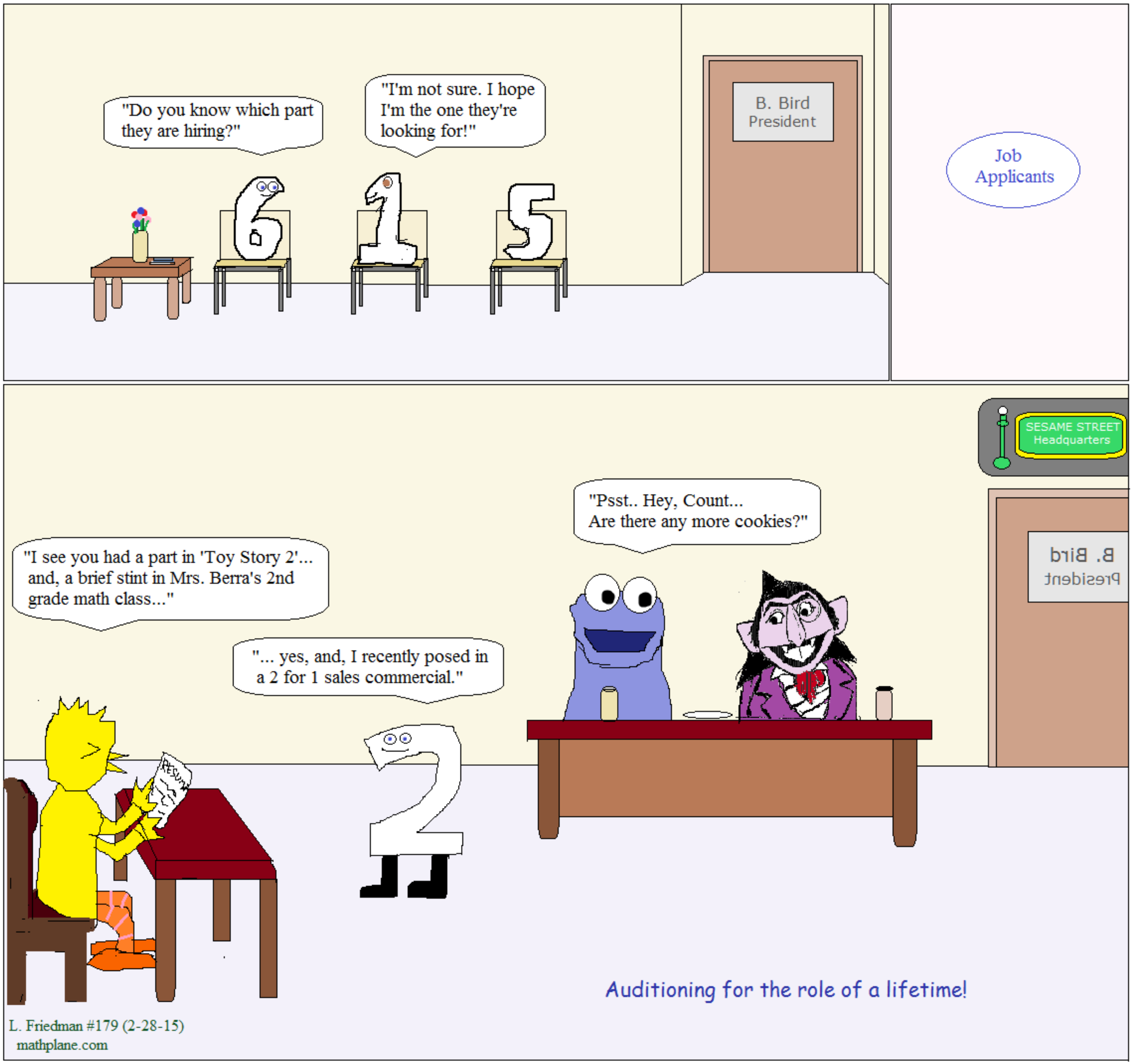
$$x^2 - 100x + 2304 = 0$$

$x =$ ~~36~~ or 64...

Distance from beach spot to snack bar is 64,

because $64^2 + 48^2 = 80^2$



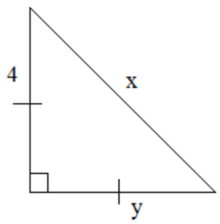


Practice Exercises-→

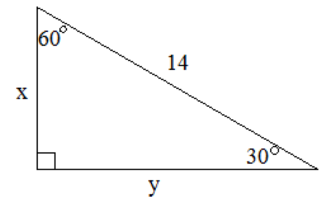
Special Right Triangles

In each triangle, find x and y. (calculator is NOT necessary)

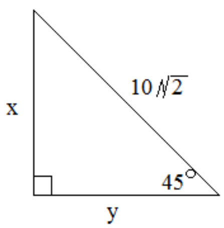
A)



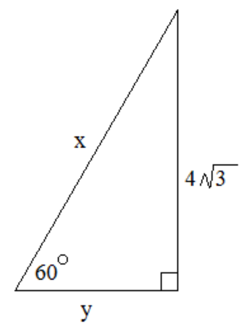
B)



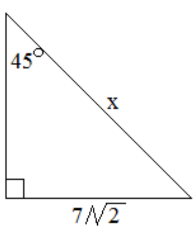
C)



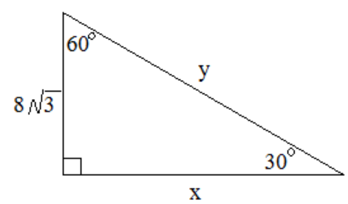
D)



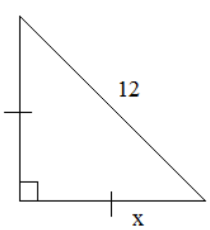
E)



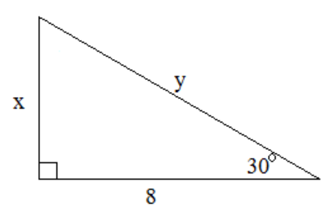
F)



G)

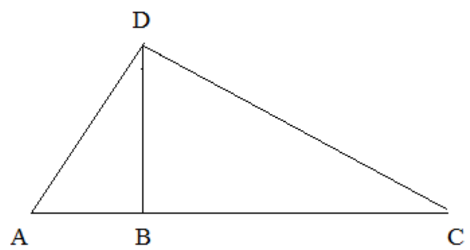


H)



Altitude, Mean Proportion, Right Triangle Review

1)



$\overline{DB} \perp \overline{AC}$

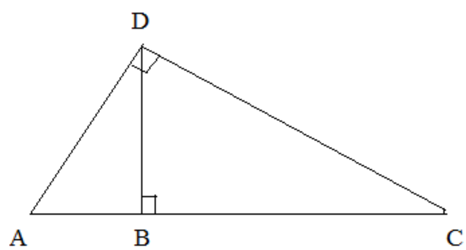
$\overline{AD} \perp \overline{CD}$

$\overline{BC} = 5$

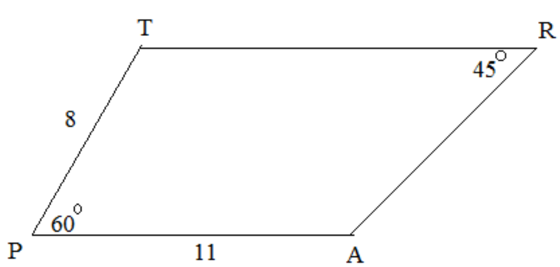
$\overline{AD} = 6$

Find the length $\overline{DB}$ and $\overline{AB}$

2) Write a similarity statement for the 3 triangles:



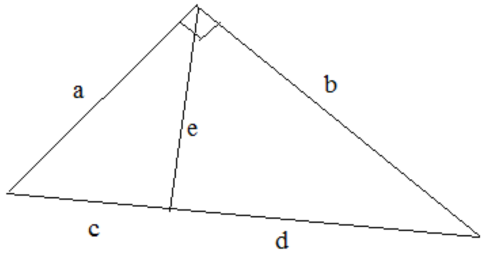
3)



Given Trapezoid TRAP, with bases $\overline{TR}$ and $\overline{PA}$...

Find $\overline{TR}$ and $\overline{RA}$

4)

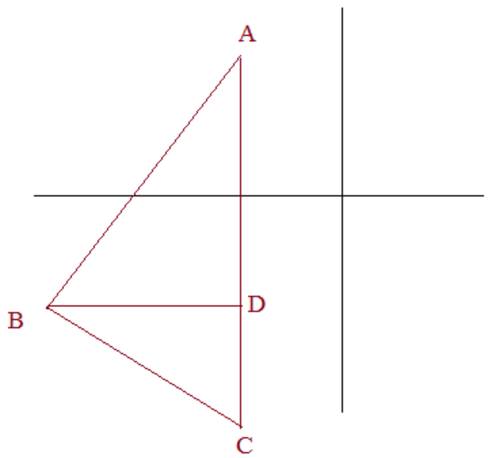


Always, Sometimes, or Never?

i) $a^2 + b^2 = (c + d)^2$

ii) $e^2 = cd$

5)



$\overline{AC} \parallel$ to the y-axis

$\overline{AC} \perp BD$ A (-4, 3) B (-10, -6)

$\overline{AB} \perp BC$

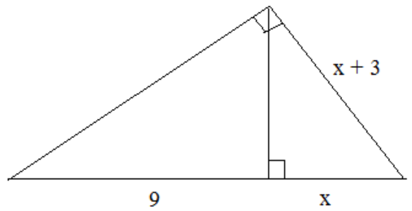
What is the coordinate of D?

What is the coordinate of C?

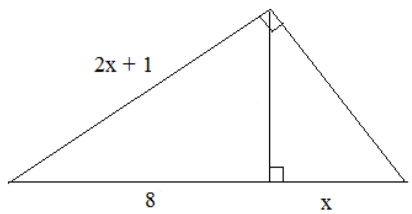
Parts of Proportional Right Triangles

Find x:

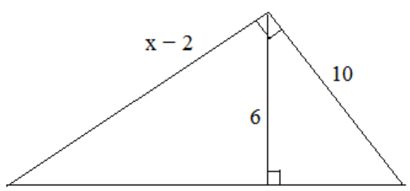
A)



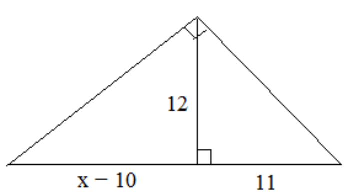
B)



C)



D)

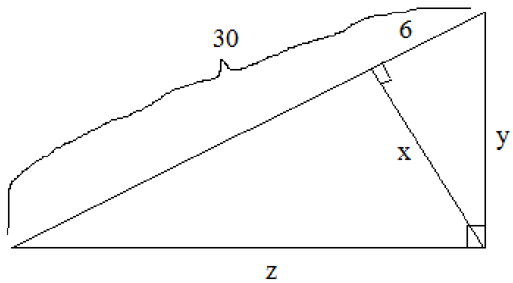


Parts of Proportional Right Triangles

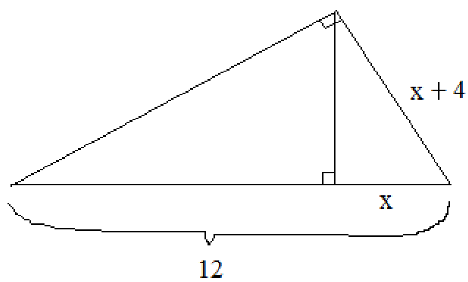
Geometric mean of divided hypotenuse is the length of the altitude

Solve:

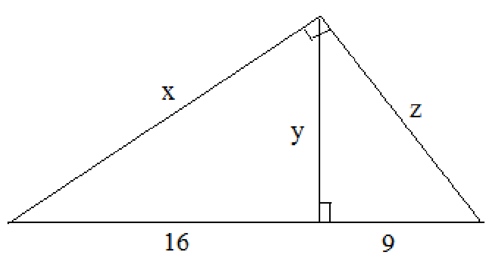
1)

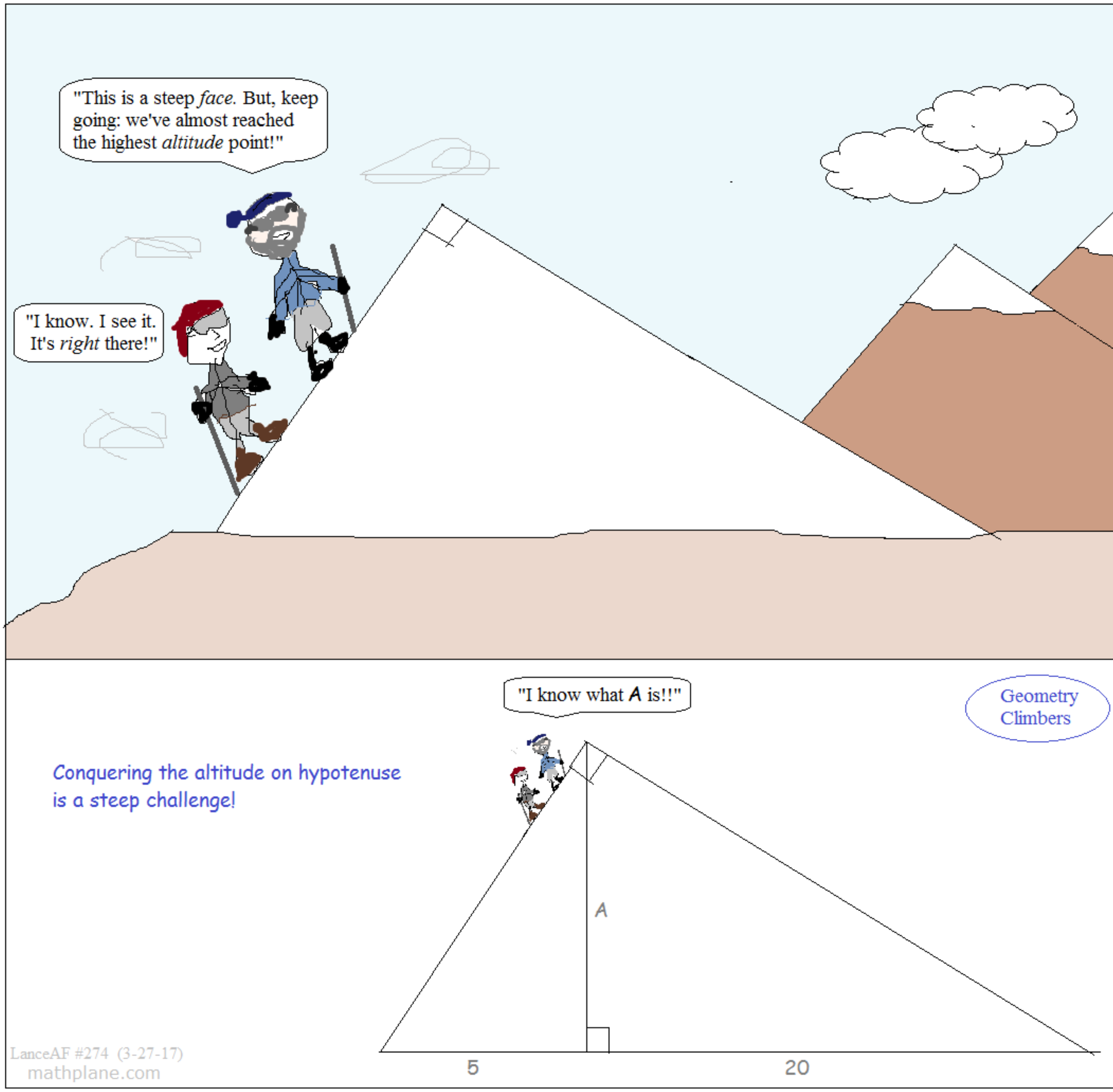


2)



3)




"This is a steep *face*. But, keep going: we've almost reached the highest *altitude* point!"
"I know. I see it. It's *right* there!"
"I know what A is!!"
Geometry Climbers
Conquering the altitude on hypotenuse is a steep challenge!
A
5
20
LanceAF #274 (3-27-17)
mathplane.com

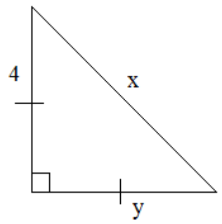
Solutions→

Special Right Triangles

SOLUTIONS

In each triangle, find x and y. (calculator is NOT necessary)

A)

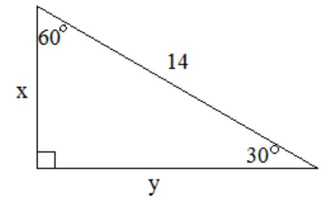


2 congruent legs, so it is a 45-45-90 right triangle...

$y = 4$

$x = 4\sqrt{2}$

B)



30-60-90 right triangle...

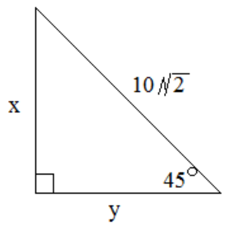
small leg is 1/2 the hypotenuse..

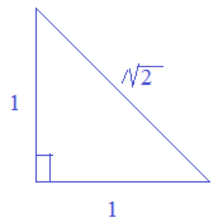
$x = 7$

medium side is small $\cdot \sqrt{3}$

$y = 7\sqrt{3}$

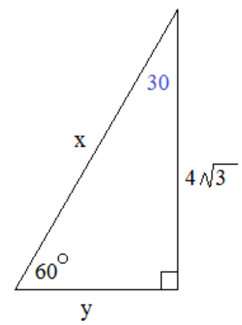
C)

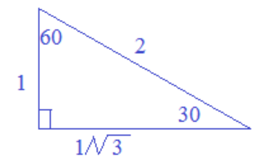




$x = 10 \qquad y = 10$

D)

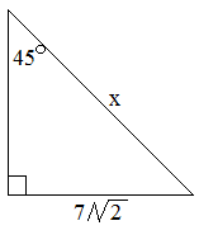




recognizing the ratios of the sides,

$y = 4$ and $x = 8$

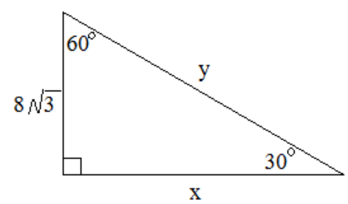
E)



$7\sqrt{2} \cdot \sqrt{2} = 14$

$x = 14$

F)



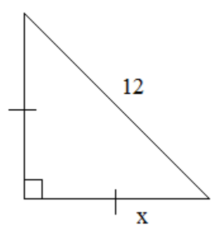
since the small leg is $8\sqrt{3}$,

the big leg is $\sqrt{3} \cdot 8\sqrt{3} = 24 = x$

and, the hypotenuse is

$2 \cdot 8\sqrt{3} = 16\sqrt{3} = y$

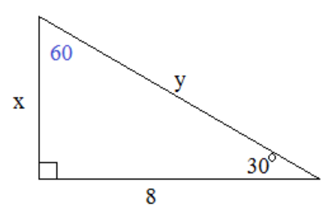
G)



$$\frac{\sqrt{2}}{1} = \frac{12}{x}$$

$$x = \frac{12}{\sqrt{2}} = 6\sqrt{2}$$

H)



$$\frac{8}{x} = \frac{\sqrt{3}}{1}$$

$$\sqrt{3}\,x = 8$$

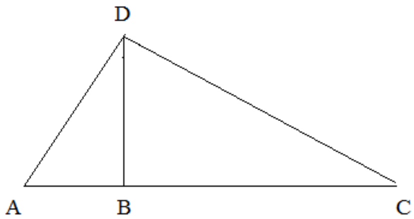
$$x = \frac{8}{\sqrt{3}}$$

$$y = 2 \cdot \frac{8}{\sqrt{3}} = \frac{16}{\sqrt{3}}$$

Altitude, Mean Proportion, Right Triangle Review

SOLUTIONS

1)



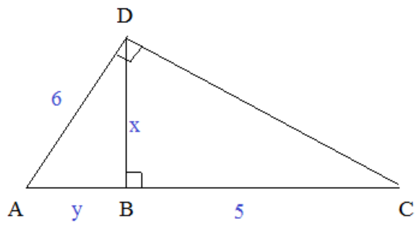
$\overline{DB} \perp \overline{AC}$

$\overline{AD} \perp \overline{CD}$

$\overline{BC} = 5$

$\overline{AD} = 6$

Find the length $\overline{DB}$ and $\overline{AB}$



$x^2 + y^2 = 36$ (Pythagorean Theorem)

$\frac{y}{x} = \frac{x}{5}$ $\frac{\text{"left/small leg"}}{\text{"bottom/large leg"}}$ Similar triangles

$x^2 = 5y$

$5y + y^2 = 36$

$y^2 + 5y - 36 = 0$

$(y + 9)(y - 4) = 0$

$y = 4$ (but, not -9 --- distance cannot be negative!)

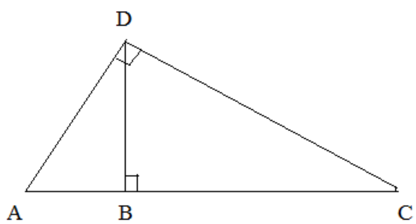
Since $y = 4$,

$x = \sqrt{20} = 2\sqrt{5}$

$\overline{DB} = 2\sqrt{5}$

$\overline{AB} = 4$

2) Write a similarity statement for the 3 triangles:

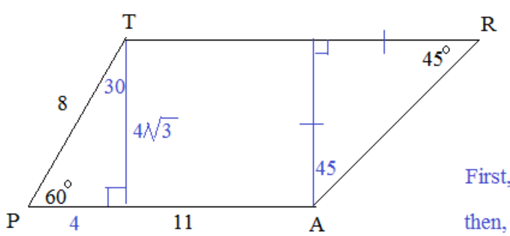


$\triangle ABD \sim \triangle DBC \sim \triangle ADC$

The similar triangles must correspond!

ex: $\triangle ABD$ is not similar to $\triangle CBD$

3)

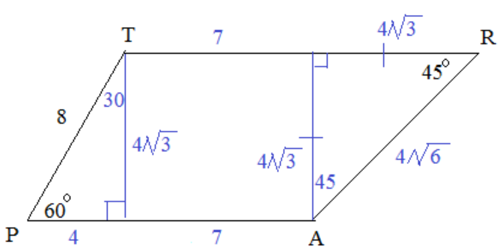


Given Trapezoid TRAP, with bases $\overline{TR}$ and $\overline{PA}$...

Find $\overline{TR}$ and $\overline{RA}$

First, draw altitudes to create right triangles..

then, using geometry properties, label the other parts..



45-45-90 rt triangle $1 : 1 : \sqrt{2}$

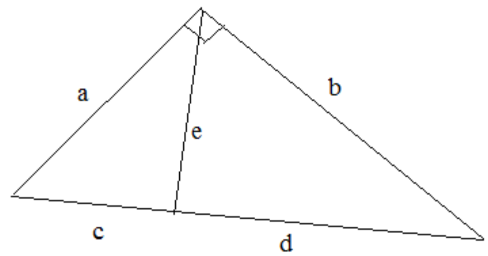
30-60-90 rt triangle $1 : \sqrt{3} : 2$

$\overline{TR} = 7 + 4\sqrt{3}$

$\overline{RA} = 4\sqrt{6}$

SOLUTIONS

4)



Always, Sometimes, or Never?

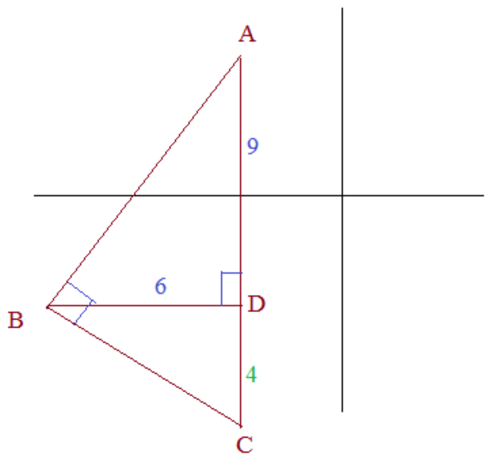
i) $a^2 + b^2 = (c + d)^2$

Always (Pythagorean Theorem)

ii) $e^2 = cd$

Sometimes (If e is an altitude, then yes.. Otherwise, no...)

5)



$\overline{AC} \parallel$ to the y-axis

$\overline{AC} \perp BD$ A (-4, 3) B (-10, -6)

$\overline{AB} \perp BC$

What is the coordinate of D? (-4, -6)

What is the coordinate of C?

AD = 9

BD = 6

Using Altitude on Hypotenuse Theorem,

$AD \cdot DC = BD^2$

$9 \quad DC = 6^2$

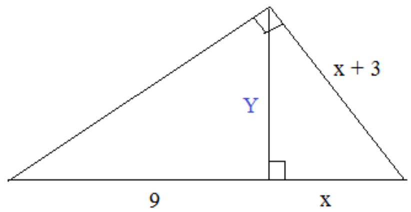
DC = 4

Therefore, point C is (-4, -10)

Parts of Proportional Right Triangles

Find x:

A)



SOLUTIONS

$Y = \sqrt{9x}$ (altitude is geometric mean of split hypotenuse)

$Y = \sqrt{(x+3)^2 - x^2}$ (Pythagorean Theorem)

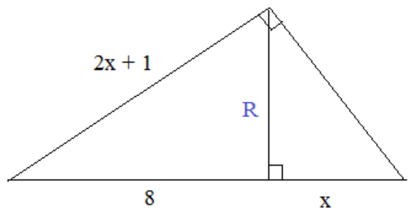
$\sqrt{9x} = \sqrt{(x+3)^2 - x^2}$ substitution

$9x = x^2 + 6x + 9 - x^2$

$3x = 9$

$x = 3$

B)



$R^2 + 8^2 = (2x+1)^2$ Pythagorean Theorem

$R^2 = 8x$ Geometric mean of altitude $\frac{8}{R} = \frac{R}{x}$

Set equations equal to each other:

$(2x+1)^2 - 8^2 = 8x$

$4x^2 + 4x + 1 - 64 = 8x$

$4x^2 - 4x - 63 = 0$

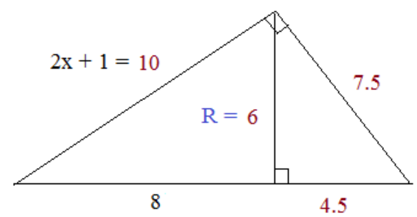
$(2x-9)(2x+7) = 0$

$x = 9/2$ or $-7/2$

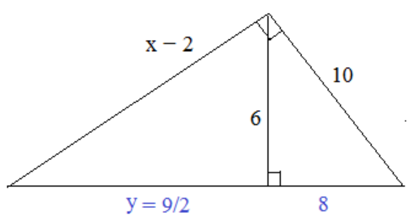
Since x cannot be negative, the solution is

$x =$ 9/2 or 4.5

To check: See if all the right triangle measures are OK



C)



1) Pythagorean Thm: $6^2 + 8^2 = 10^2$

2) Alt. to Hypotenuse: 6 is geometric mean of y and 8

$6^2 = 8y$ then, $y = 36/8 = 9/2$

3) Pythagorean Thm: $6^2 + (9/2)^2 = (x-2)^2$

$36 + 81/4 = x^2 - 4x + 4$

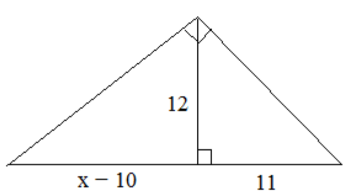
$x^2 - 4x - 52.25 = 0$

$x = 9.5$ or -5.5 (quadratic formula)

Since a side cannot be negative, $x = 9.5$

To check: observe all the right triangles:
6-8-10 4.5-6-7.5 7.5-10-12.5
2 x (3-4-5) 1.5 x (3-4-5) 2.5 x (3-4-5)

D)



Altitude to hypotenuse: $12^2 = 11(x - 10)$

$144 = 11x - 110$

$11x = 254$

$x = 23.1$

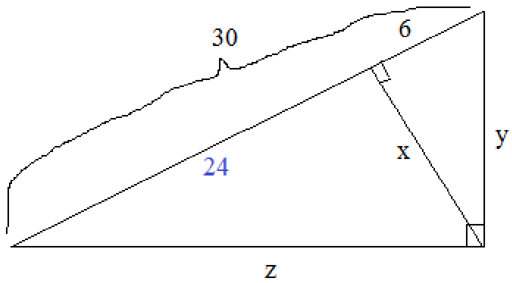
Parts of Proportional Right Triangles

Geometric mean of divided hypotenuse is the length of the altitude

Solve:

SOLUTIONS

1)



$x^2 = (24)(6)$

$x = 12$

Altitude to Hypotenuse Theorem

$x^2 + 6^2 = y^2$

$144 + 36 = y^2$

$y = \sqrt{180} = 6\sqrt{5}$

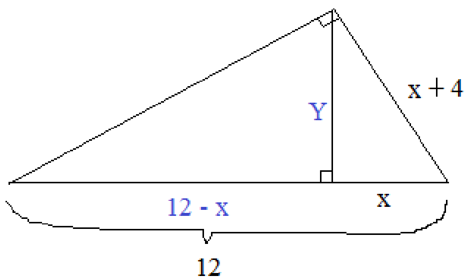
Pythagorean Theorem

$y^2 + z^2 = 30^2$

$180 + z^2 = 900$

$z = \sqrt{720} = 12\sqrt{5}$

2)



$Y^2 = (x + 4)^2 - x^2$ Pythagorean Theorem

$Y^2 = (x)(12 - x)$ Altitude to Hypotenuse Theorem

(Substitution): set equations equal to each other

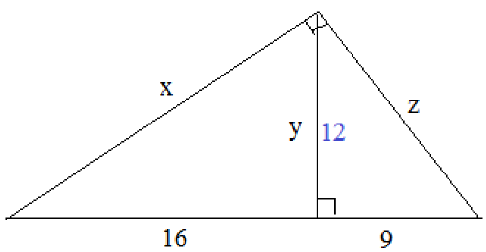
$(x + 4)^2 - x^2 = (x)(12 - x)$

$8x + 16 = 12x - x^2$

$x^2 - 4x + 16 = 0$

NO SOLUTION!!

3)



$\frac{y}{9} = \frac{16}{y}$

$y = 12$

Altitude to Hypotenuse Theorem

$z = 15$

Pythagorean Triple

3 x (3-4-5) = 9-12-15 right triangle

$x^2 + z^2 = 25^2$

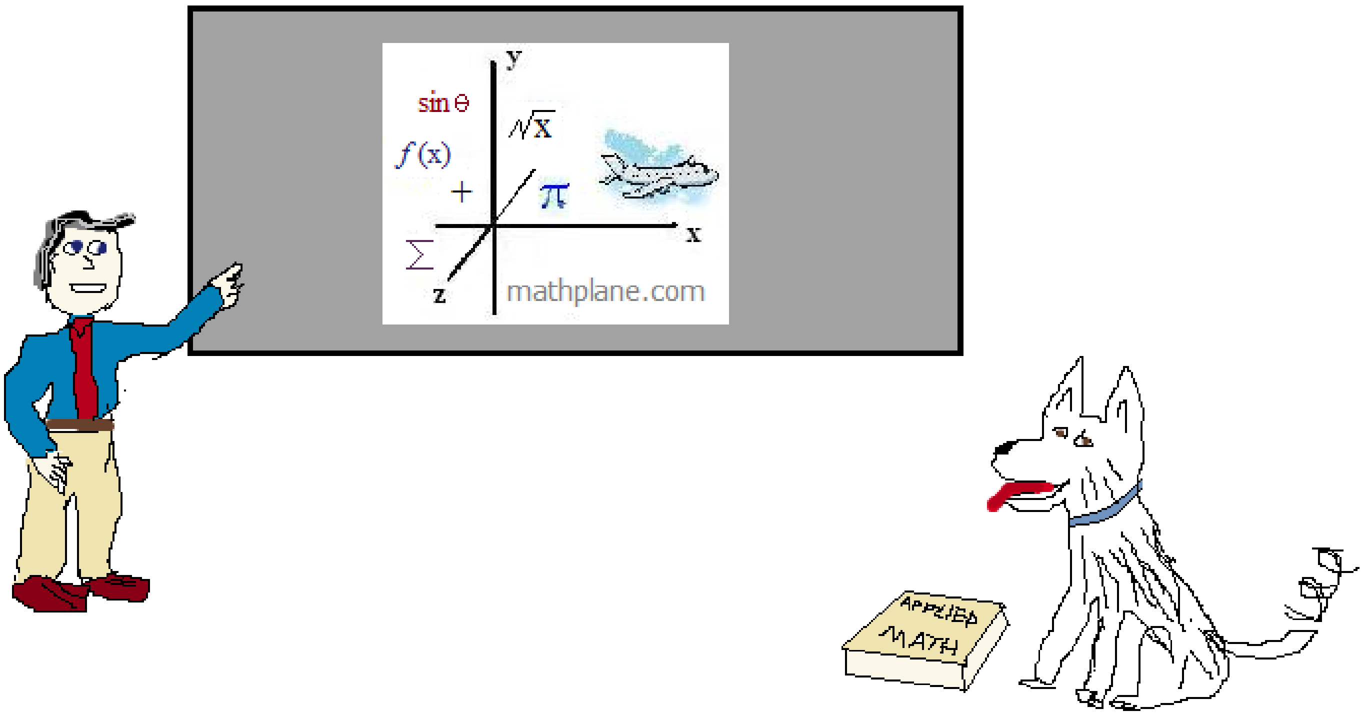
$x^2 + 225 = 625$

$x = 20$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Enjoy



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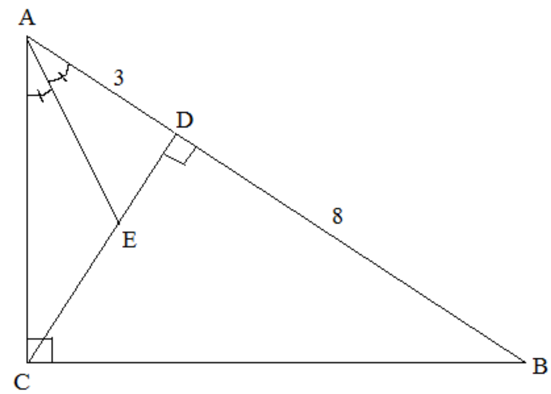
One more example -→

Angle Bisector, Pythagorean Theorem, and Means/Proportional

Find the length of $\overline{DE}$



Step 1: Utilize the "Geometric Mean of divided Hypotenuse"

$$\frac{AD}{DC} = \frac{DC}{DB}$$

$$DC^2 = AD \cdot DB$$

$$DC = \sqrt{24}$$

Step 2: Utilize the Pythagorean Theorem

$$DB^2 + DC^2 = CB^2$$

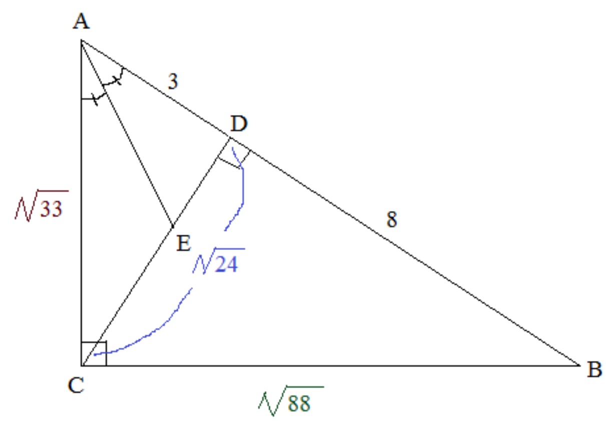
$$64 + 24 = CB^2$$

$$CB = \sqrt{88}$$

$$CB^2 + AC^2 = AB^2$$

$$88 + AC^2 = 121$$

$$AC = \sqrt{33}$$



Step 3: Use the "Angle Bisector Theorem"

Since AE is an angle bisector in triangle CAD,

$$\frac{AD}{AC} = \frac{DE}{CE}$$

$$\frac{3}{\sqrt{33}} = \frac{x}{\sqrt{24} - x}$$

$$3\sqrt{24} - 3x = \sqrt{33}\,x$$

$$3\sqrt{24} = \sqrt{33}\,x + 3x$$

$$14.697 = 8.745x$$

$$\boxed{x = 1.68}$$