Random (Math) Places to Visit

Topics for discussion include Clock Angles, Multiplication, Doubling, Linear vs. Exponential Equations, Area, Vinculum Bar, and more.

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Million Dollar Question: 1 million dollars OR 1 penny doubled daily for one month?

A man offers each son the following choice: You may have 1 million dollars or *one penny doubled every day for a month.* Which one would you choose?

The first son looks at the briefcase filled with $1,000,000. Then, he looks at the penny his father is holding. He does a quick "guestimate":

.01 .02 .04 .08 .16 .32 .64 1.28 2.56 5.12 10.24

He pauses, and thinks, 'so one penny roughly reaches $10 after ten "doubles"

10 20 40 80 160 320 640 1,280 2,560 5,120 10,240

He pauses, and realizes, '$10 will reach over $10,000 after ten more "doubles"'

And, then $10K will reach over $1,000,000 after ten more "doubles"

The first son accepts the penny, knowing it'll double easily over $1 million in 30 days..

1st calculation: "Guestimate" and 'rounding down'..

The second son pulls out a calculator to find the exact amount.

He determines the equation is the following: $.01 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 \cdot 2 \cdot 2 = .01 \times 2^n$

where n is the number of times the penny doubles...

He asks his father, "Which month? February is only 28 days." The father replies, "the particular month doesn't affect your decision, does it?"

After punching in the numbers, the son realizes the penny doubles over $1,000,000 on the 28th day."

2nd calculation: Exact amount
**notice the difference when there is no rounding down (from 10.24 to 10 and from 10,240 to 10,000)!!

day 1: 1 penny
day 2: 2 cents
day 3: 4 cents
...
day 25: $.01 \times 2^{24}$ = $167,772.16
day 26: $.01 \times 2^{25}$ = $335,544.32
day 27: $.01 \times 2^{26}$ = $671,088.64
day 28: $.01 \times 2^{27}$ = $1,342,177.28
day 29: $.01 \times 2^{28}$ = $2,684,354.56
day 30: $.01 \times 2^{29}$ = $5,368,709.12
day 31: $.01 \times 2^{30}$ = $10,737,418.24

The third son immediately accepts the offer and adds, "Yes, double the amount each day and I'll add it to this bank account." The father quickly responds, "Wait, that'll cost me almost double the amount!!"

(Did it?)

3rd calculation: Cumulative amount

day 1: 1 penny
day 2: add 2 cents (total 3 cents)
day 3: add 4 cents (total 7 cents)
day 4: add 8 cents (total 15 cents)
day 5: add 16 cents (total 31 cents)
day 6: add 32 cents (total 63 cents)

(Can you see the pattern?)

$$\sum_{n=1}^{31} .01 \times 2^{n-1}$$

$$\sum_{n=0}^{30} .01 \times 2^{n} = .01 \sum_{n=0}^{30} 2^{n} = .01\left(\frac{2^{31}-1}{2-1}\right) = 21{,}474{,}836.47$$

(1 penny less than double the amount)

"Average Speed of a Round-Trip"

Suppose you drive a car from your house to the airport at a speed of 20 miles per hour. Then, you pick up your friend, turn around and drive home at a speed of 30 miles per hour. What was the average speed of the total round-trip?

> Incorrect Solution: A quick guess would be 25 miles per hour. After all, the distance is the same, and the average of 20mph and 30mph is 25 miles per hour. ✗

Let's check: Suppose the airport is 60 miles away (you're very nice for driving 60 miles to pick up your friend!)

distance = rate x time

(to the airport)
60 miles = 20 miles/hour x time
time = 3 hours

(back home)
60 miles = 30 miles/hour x time
time = 2 hours

⇨ Total time for the round-trip: 5 hours (there was a lot of traffic!)

Total distance for the round-trip: 120 miles

$d = rt$

120 miles = r (5 hours)
r = 24 miles/hour ✓

What happened? More time is spent going 20 mph than going 30 mph. Therefore, this has more influence, lowering the overall average.

General Solution:

(one-way)

$\text{distance}_1 = \text{rate}_1 \times \text{time}_1$

$d_1 = \frac{20\text{miles}}{\text{hour}} \times t_1$

$\text{time}_1 = \frac{d}{20 \text{ m/h}}$

(return)

$\text{distance}_2 = \text{rate}_2 \times \text{time}_2$

$d_2 = \frac{30\text{miles}}{\text{hour}} \times t_2$

$\text{time}_2 = \frac{d}{30 \text{ m/h}}$

(total round-trip)

$d_1 + d_2 = \text{rate} \times \text{time}_{total}$

Since the distances are equal,

$2d = \text{rate} \times \text{time}_{total}$

Total round-trip:

$$2d = \text{rate (time 1 + time 2)}$$

$$2d = \text{rate}\left(\frac{d}{20 \text{ m/h}} + \frac{d}{30 \text{ m/h}}\right)$$

$$2d = \text{rate}\left(\frac{3d}{60 \text{ m/h}} + \frac{2d}{60 \text{ m/h}}\right)$$

$$2d = \text{rate}\left(\frac{1d}{12 \text{ m/h}}\right)$$

> 24 m/h = rate

There is a common math problem thata goes like this:

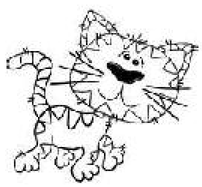
> There is a bus with 7 girls inside. Each girl has 7 backpacks.
> Inside each backpack, there are 7 big cats.
> Each big cat has 7 small cats.
> How many legs are on the bus?

**Two comments:

1) There is no bus driver on the bus..
2) As a riddle, one may say the answer is zero because there are no "legs on a bus" (only wheels)..

But, let's assume this is a standard math problem. What is the answer?

(**Final count is in the lower right corner)

Solution:
(read the question carefully and count the legs)

There are 7 girls... 2 x 7 = 14 human legs

Each girl has 7 backpacks.. 7 x 7 = 49 backpacks

ONE backpack has 7 big cats.. 4 x 7 = 28 big cat legs
Each big cat has 7 small cats..
7 x 7 = 49 small cats.. 4 x 49 = 196 small cat legs

So, ONE backpack has 224 total cat legs..

And, there are 49 backpacks..
So, the total number of cat legs are 49 x 224 = 10976

Therefore, the total number of legs on the bus

Total number of legs: 10976
+ 14
10990

Vinculum (Bar) and "Repeating Decimals"

Vinculum: A horizontal bar drawn over multiple quantities to indicate they are grouped together.

Examples include:

- radicals $\sqrt{9X^3}$
- line segments $\overline{AB}$ (joining points A & B)
- repeating decimals $0.77\overline{76}$

Repeating Decimals: A decimal number that eventually becomes periodic (i.e. "the end repeats indefinitely")

Examples:

$$\frac{1}{3} = 0.333333... = 0.\overline{3}$$

$$\frac{22}{7} = 3.\underbrace{142857}\underbrace{142857}... = 3.\overline{142857}$$

$$12.0340353535... = 12.0340\overline{35}$$

Converting Fractions to Decimals: Divide the numerator by the denominator

Examples:

$\frac{42}{9}$

$$\begin{array}{r} 4.66\overline{6} \\ 9\,\overline{)\,42} \\ -36 \\ \hline 60 \\ -54 \\ \hline 60 \\ -54 \\ \hline \text{etc....} \end{array}$$

$4.\overline{6}$

$\frac{3}{700}$

$$\begin{array}{r} 0.00428571 \\ 700\,\overline{)\,3.0000000} \\ -2.800 \\ \hline .2000 \\ -.1400 \\ \hline 6000 \\ -5600 \\ \hline 4000 \\ -3500 \\ \hline 5000 \\ -4900 \\ \hline 1000 \\ -700 \\ \hline 3000 \end{array}$$

etc... (repeats indefinitely)

$0.00\overline{428571}$

Converting 'Repeating Decimals' to Fractions: Using algebra

Examples: $.\overline{7}$ let $n = .7\overline{7}$ then, $10n = 7.7\overline{7}$

$$\begin{array}{r} 10n \\ -\ \ n \\ \hline 9n \end{array} \qquad \begin{array}{r} 7.7\overline{7} \\ -.7\overline{7} \\ \hline 7.0 \end{array}$$

substitution reveals that $9n = 7$ $\quad n = \frac{7}{9}$

$11.\overline{18}$ let $m = .\overline{18}$ then, $100m = 18.\overline{18}$

$$\begin{array}{r} 100m \\ -\ \ m \\ \hline 99m \end{array} \qquad \begin{array}{r} 18.\overline{18} \\ -\ .\overline{18} \\ \hline 18 \end{array}$$

$11.\overline{18} = 11 + m$

$m = \frac{18}{99}$

$11.\overline{18} = 11\frac{18}{99}$

$234.00176\overline{76}$

Separate the number into parts: $234 + .001 + .00\overline{76}$

Convert the parts to fractions: $234 + .001 = \frac{1}{1000} = \frac{99}{99000}$ + let $p = .00076\overline{76}$

$1000p = .76\overline{76}$

$100000p = 76.\overline{76}$

$$\begin{array}{r} 100000p \\ -\ 1000p \\ \hline 99000p \end{array} \qquad \begin{array}{r} 76.\overline{76} \\ -\ .\overline{76} \\ \hline 76 \end{array}$$

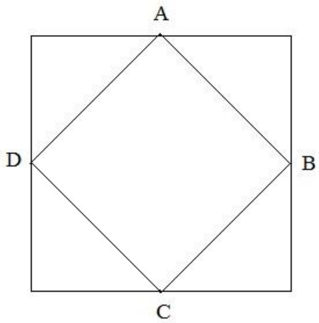
$p = \frac{76}{99000}$

$.00\overline{76} = \frac{76}{99000}$

Combine the Fractions: $234 + \frac{99}{99000} + \frac{76}{99000} = 234\frac{175}{99000}$

"Area of a square inscribed in a square"

Square ABCD is inscribed in the larger square.
If the area of the larger square is 100 sq. feet, what is the area of ABCD?

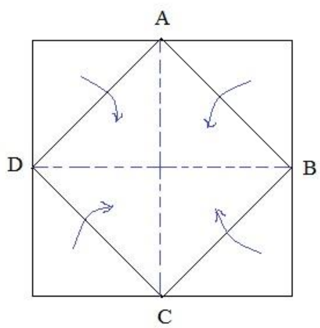


Note: There are a few approaches to solving.

Hint: Since it is a square inscribed in a square, A, B, C, and D are midpoints of the sides of the larger square!

Solution using different approaches:

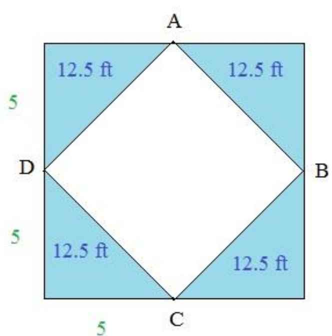
1) Observations and recognizing triangles



The large triangle contains 8 congruent triangles
and
the small triangle contains 4 of those triangles..

Therefore, the area is 4/8 (100) = 50 square feet!

2) large square $-$ extra triangles = inside square



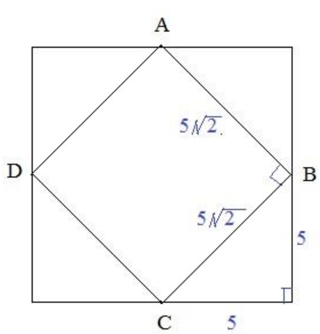
Area of large square = 100 sq. feet
Therefore, each side is 10 feet

Since A, B, C, and D are midpoints, each triangle has a base of 5 and height of 5..
Therefore, the area of each triangle is 1/2bh = 25/2

Since there are 4 triangles, the extra area is 50 sq. feet.

So, the inside square is 100 feet - 50 feet = 50 feet...

3) Calculate the area of the inside square



Consider the right triangle (45-45-90)
From this (or the pythagorean theorem),
we know the length of each side of the inside square is $5\sqrt{2}$

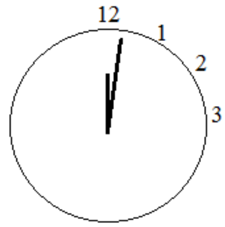
Area of a square is S^2

So, area of inscribed square is $\left(5\sqrt{2}\right)^2 =$ 50 square feet!

Question:

A clock sits at exactly 12:02...

What time will the minute hand cross the hour hand?
Express the answer to the nearest second...



Measurements to remember:

60 minutes = 1 hour

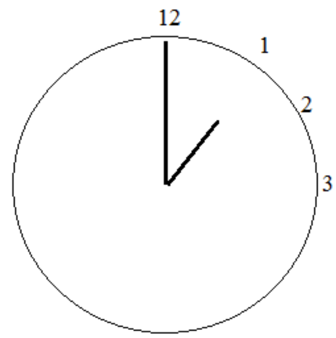
60 seconds = 1 minute

3600 seconds = 1 hour

Solution:

The hour hand will move from one number to the next every 60 minutes (or 3600 seconds)

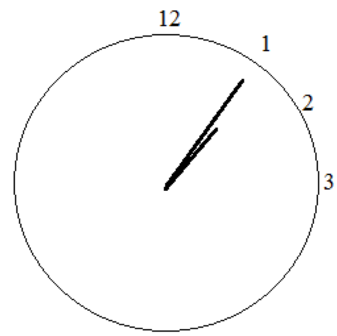
The minute hand will move from one number to the next every 5 minutes (or 300 seconds)



1:00

The minute hand goes around the clock and returns to the top. (1:00)... The minute hand is on the 12, and the hour hand is on the 1....

Then, the minute hand reaches the 1 five minutes later (1:05)...

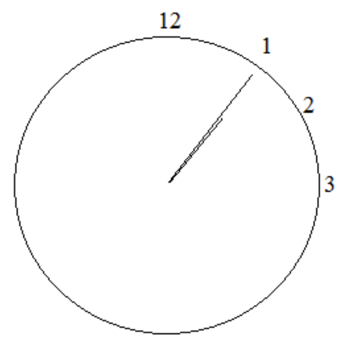


1:05

***But, during those 5 minutes, the hour hand moved!!

How far?

The hour hand moves from 1 to 2 in 60 minutes...
So, during the 5 minutes, the hour hand moved 5/60 or 1/12 of the way to 2...
So, how long would it take for the minute hand to travel 1/12 of the way to 2?
Well, it takes 300 seconds for the minute hand to travel from 1 to 2...
Therefore, it takes 25 seconds to travel 1/12 of the way!!

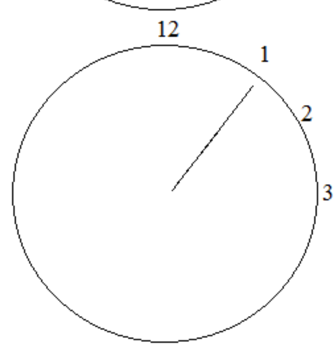


1:05:25

But, wait.... During those 25 seconds, the hour hand is still moving...
So, during the 25 seconds, how far did the hour hand move?

Well, it takes 3600 seconds for the hour hand to move from 1 to 2...
Therefore, in 25 seconds, the hour hand moved 25/3600 of the distance.. This is approx. .007

Again, it takes 300 seconds for the minute hand to travel from 1 to 2.
Therefore, it would take approx. 2 seconds to travel .007 of the way between 1 and 2.



1:05:27

So, the minute hand has moved
63 minutes to get to 1:05...
Then, 25 seconds to close in on the hour hand... 1:05:25...
And, finally, 2 more seconds to reach the hour hand... 1:05:27

Linear vs. Exponential Model

Example: Given the coordinates (0, 100) and (2, 400)

Find a *linear* equation that passes through the coordinates.

Find an *exponential* equation that goes through the coordinates.

Graph both functions.

To find a linear equation, you need the slope and a point.

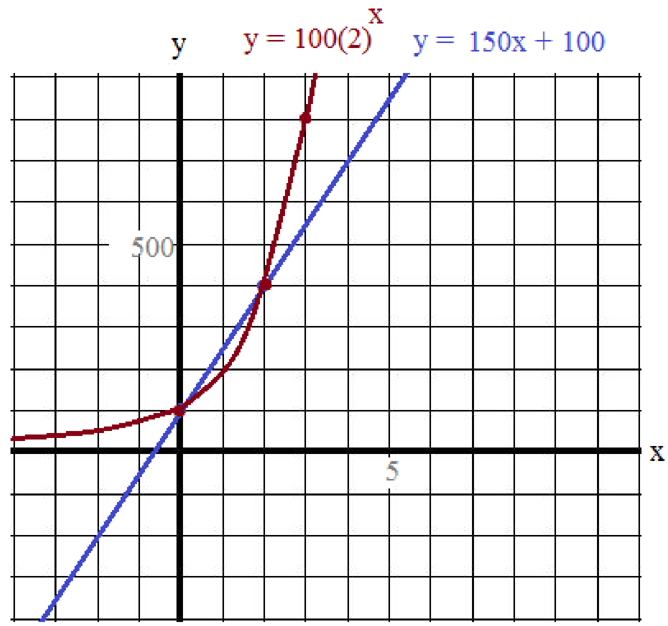
Slope:

$$\frac{\text{"rise"}}{\text{"run"}} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{400 - 100}{2 - 0} = 150$$

A point: (0, 100)

Equation of the line: $y - 100 = 150(x - 0)$

or $y = 150x + 100$



To find the exponential equation, use the general form $y = ab^x$

Substitute (0, 100): $100 = ab^0$

$100 = a(1)$

$a = 100$

Substitute (2, 400): $400 = ab^2$

$400 = (100)b^2$

$b^2 = 4$

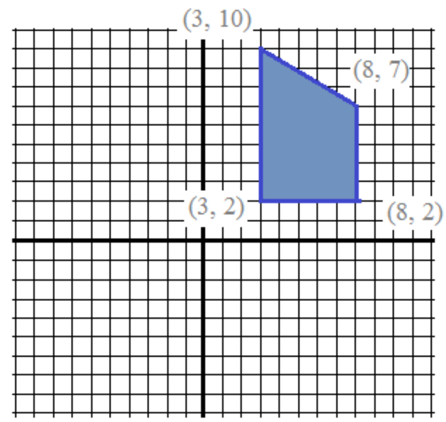
$b = 2$

Equation of the exponential curve: $y = 100(2)^x$

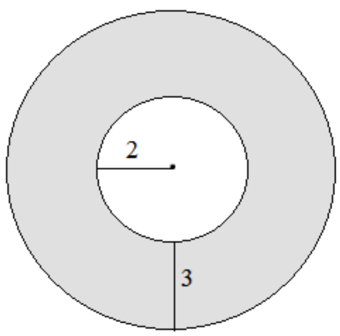
Three Area Problems….

What is the area of each shaded region?

1) Quadrilateral

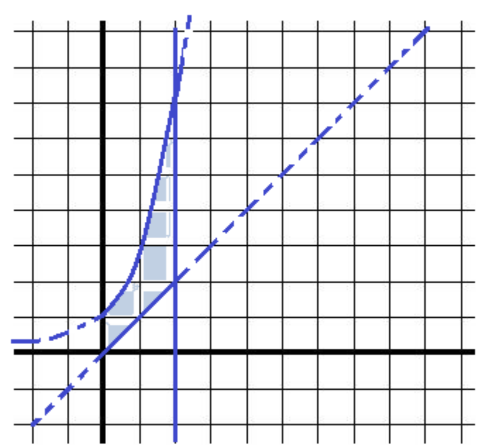


2) Concentric Circles



3) The area bordered by:

$y = e^{x}$
$y = x$
$x = 2$
the y-axis



Three Area Problems....

What is the area of each shaded region?

1) Quadrilateral

method 1:
square + triangle

(side x side) $\frac{1}{2}$ (base)(height)

$(5 \times 5) + \frac{1}{2}(5)(3)$

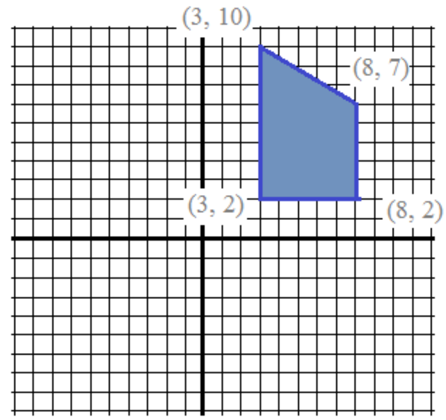
$25 + 15/2 = 32.5$

method 2:
(sideways) trapezoid

$\frac{1}{2}$(base1 + base2)(height)

$\frac{1}{2}(8 + 5)(5)$

$\frac{1}{2}\,65 = 32.5$



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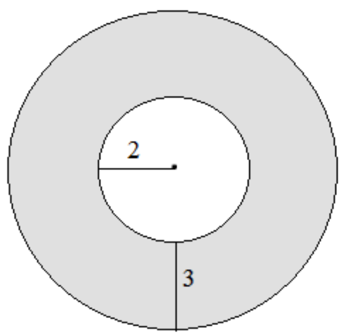


radius of big circle: $2 + 3 = 5$
area of big circle: $\pi r^2 = 25\pi$

radius of small circle: 2
area of small circle: $\pi r^2 = 4\pi$

area of shaded region = $\text{area}_{big} + \text{area}_{small}$

$25\pi - 4\pi = 21\pi$

3) $y = e^x$
$y = x$
$x = 2$
the y-axis

$y = e^x$

x	y
-1	.37
0	1
1	2.72
2	7.39

(approx.)

The shaded area consists of the area under the log function MINUS the right triangle (i.e. the area under the line y = x)

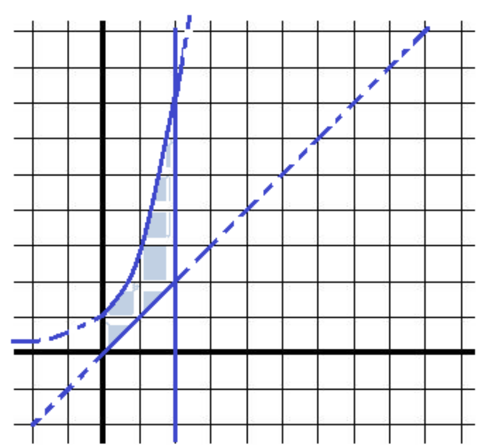
(use definite integral to find area)

$A_{log} = \int_0^2 e^x\,dx = e^2 - e^0$

$= 7.39 - 1$

$A_{tri} = \frac{1}{2}(2)(2) = 2$

Total Area is approx. 4.39 sq. units



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Enjoy,

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