Algebra 2: Math Induction

Notes, Examples, and Practice Exercises (with Solutions)

Topics include factoring, sigma notation, exponents, factorials, sequences and series, and more.

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Example: Use induction to prove $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Step 1: Verify it works for $n = 1$ (and, perhaps, a few others)

If $n = 1$ $\quad 1 = \frac{1(1+1)}{2} = 1$

If $n = 2$ $\quad 1 + 2 = \frac{2(2+1)}{2} = 3$

If $n = 3$ $\quad 1 + 2 + 3 = \frac{3(3+1)}{2} = 6$

So, the equation works for these numbers... How do we know if it works for <u>any</u> integer?!?

Step 2: Assume the formula is correct. Then, evaluate for the <u>next</u> term...

If the formula is correct, then $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

Then, if we evaluate the next term, $1 + 2 + 3 + \ldots + n + (n+1) = \frac{(n+1)((n+1)+1)}{2}$ or $\frac{(n+1)(n+2)}{2}$

So, how do we confirm this assumption?!?!

Step 3: Using our confirmed equations, add the next term..

When we tested $n = 1, 2$ and 3, we had the correct solution...

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Let's add the next term to both sides...

$$1 + 2 + 3 + \ldots + n + (n+1) = \frac{n(n+1)}{2} + (n+1)$$

$$= \frac{n(n+1)}{2} + \frac{2(n+1)}{2}$$

$$= \frac{n(n+1) + 2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$(n+1) + \sum_{k=1}^{n} k = \frac{n(n+1)}{2} + (n+1)$$

$$= \frac{n(n+1)}{2} + \frac{2(n+1)}{2}$$

$$= \frac{n(n+1) + 2(n+1)}{2}$$

$$\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$$

Step 4: Compare and conclusion

Notice, in step 2, when we directly input $(n+1)$ into the formula, we get $\frac{(n+1)(n+2)}{2}$

Then, in step 3, when we added the next term to previous known terms, we get $\frac{(n+1)(n+2)}{2}$

Since both approaches lead to the same answer, the general formula works for every successive integer!

Example: Use induction to prove $\sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{n(n+1)(n+2)}{6}$ or, $1 + 3 + 6 + ... + \frac{k(k+1)}{2} = \frac{k(k+1)(k+2)}{6}$

Step 1: Verify it works for $n = 1$ (and, perhaps, a few others)

If $n = 1$ $\quad 1 = \frac{1(1+1)(1+2)}{6} = 1$

If $n = 2$ $\quad 1 + 3 = \frac{2(2+1)(2+2)}{6} = 4$

If $n = 3$ $\quad 1 + 3 + 6 = \frac{3(3+1)(3+2)}{6} = 10$

So, the equation works for these numbers... How do we know if it works for any integer?!?

Step 2: Assume the formula is correct. Then, evaluate for the next term...

If the formula is correct, then $1 + 3 + 6 + ... + \frac{k(k+1)}{2} = \frac{k(k+1)(k+2)}{6}$

Then, if we evaluate the next term, $1 + 3 + 6 + ... + \frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)(k+3)}{6}$

So, how do we confirm this assumption?!?!

Step 3: Using the confirmed equations in step 1, add the next term to the confirmed basis..

When we tested $n = 1$, 2 and 3, we had the correct solution...

$$1 + 3 + 6 + ... + \frac{k(k+1)}{2} = \frac{k(k+1)(k+2)}{6}$$

Let's add the next term to both sides...

$$1 + 3 + 6 + ... + \frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2} = \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2}$$

common denominator

$$\frac{k(k+1)(k+2) + 3(k+1)(k+2)}{6}$$

GCF: $(k+1)(k+2)$

$$\frac{(k+1)(k+2) \cdot (k+3)}{6}$$

Using Summation notation:

$$\sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

$$\frac{(n+1)(n+2)}{2} + \sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{n(n+1)(n+2)}{6} + \frac{(n+1)(n+2)}{2}$$

$$= \frac{n(n+1)(n+2)}{6} + \frac{3(n+1)(n+2)}{6}$$

$$= \frac{(n+1)(n+2)(n+3)}{6}$$

$$\sum_{k=1}^{n+1} \frac{k(k+1)}{2} = \frac{(n+1)(n+2)(n+3)}{6}$$

Step 4: Compare and conclusion

Notice, in step 2, when we directly input $(k+1)$ into the formula, we get $\frac{(k+1)(k+2)(k+3)}{6}$

Then, in step 3, when we added the next term to previous known terms, we get $\frac{(k+1)(k+2)(k+3)}{6}$

Since both approaches lead to the same answer, the general formula works for every successive integer!

Example: Use induction to prove $n^3 - 4n + 6$ is a *multiple of 3*

Step 1: Verify the base case(s):

if n = 1, then $(1)^3 - 4(1) + 6 = 3$

if n = 2, then $(2)^3 - 4(2) + 6 = 6$ They are all multiples of 3...

if n = 3, then $(3)^3 - 4(3) + 6 = 21$

Step 2: Assume for k:

$k^3 - 4k + 6 = 3Z$ where Z is any integer... so, 3Z is a multiple of 3...

Step 3: Prove for next term, (k + 1)

$(k+1)^3 - 4(k+1) + 6 = (k+1)(k+1)(k+1) - 4k - 4 + 6$ expand

$(k^2 + 2k + 1)(k+1) - 4k + 2$ collect like terms

$k^3 + 3k^2 + 3k + 1 - 4k + 2$ rearrange

$k^3 - 4k + 3k^2 + 3k + 1 + 2$ regroup

$k^3 - 4k + 3k^2 + 3k + 3$ ***add multiples of 3...

$k^3 - 4k + 6 + 3k^2 + 3k + 3 + 3k$ 6 and 3k are each divisible by 3...

$3Z + 3(k^2 + 2k + 1)$ Both terms are divisible by 3 (i.e. multiples of 3)...

Note: We know 3k is divisible by 3. So, if x is divisible by 3, then x + 3k must also be divisible by 3!!

Example: $F_1 = 1$, $F_2 = 1$ $F_n = F_{n-1} + F_{n-2}$

Prove $F_n \leq 2^n$ by induction...

Step 1: Look at the base cases and verify (It's apparent that F is a Fibonacci sequence..)

$F_1 = 1 \leq 2^1$ $F_3 = 2 \leq 2^3$

$F_2 = 1 \leq 2^2$ $F_4 = 3 \leq 2^4$

$F_5 = 5 \leq 2^5$

Step 2: Assume for k

$F_k \leq 2^k$

Step 3: Prove for k + 1

$F_{k+1} \leq 2^{k+1}$ We know (by definition) that $F_{k+1} = F_k + F_{k-1}$

$\leq 2^k \cdot 2^1$ $F_k \leq 2^k$ and, $F_{k-1} \leq 2^k$

$\leq 2 \cdot 2^k$

$F_{k+1} \leq 2^k + 2^k$ Therefore, $F_k + F_{k-1} \leq 2^k + 2^k$

Example: Prove: $8^n - 3^n$ is a multiple of 5

Step 1: Verify a few cases

If $n = 1$, then 5 ✓

If $n = 2$, then $64 - 9 = 55$ ✓

Step 2: Set up Assumption: $8^k - 3^k = 5P$

Since it works for k = 1 and k = 2, prove using induction that it works for the following terms...

$$8^{k+1} - 3^{k+1} =$$

$$8^k \cdot 8^1 - 3^k \cdot 3^1 =$$

$8^k - 3^k = 5P$

$8^k = 5P + 3^k$

using substitution

$$(5P + 3^k) \cdot 8 - 3 \cdot 3^k =$$

$$40P + 8 \cdot 3^k - 3 \cdot 3^k =$$

$$40P + (8 - 3) \cdot 3^k =$$

$$40P + 5 \cdot 3^k =$$

$$5(8P + 3^k) =$$

must be multiple of 5!!! ✓

Example: Prove $\prod_{i=2}^{n} \frac{i-1}{i} = \frac{1}{n}$

Verify a few cases:

If n = 3,

$$\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$ ✓

If n = 4,

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{4}$$ ✓

We must show that

$$\prod_{i=2}^{n+1} \frac{i-1}{i} = \frac{1}{n+1}$$

Next term:

first n terms, last term

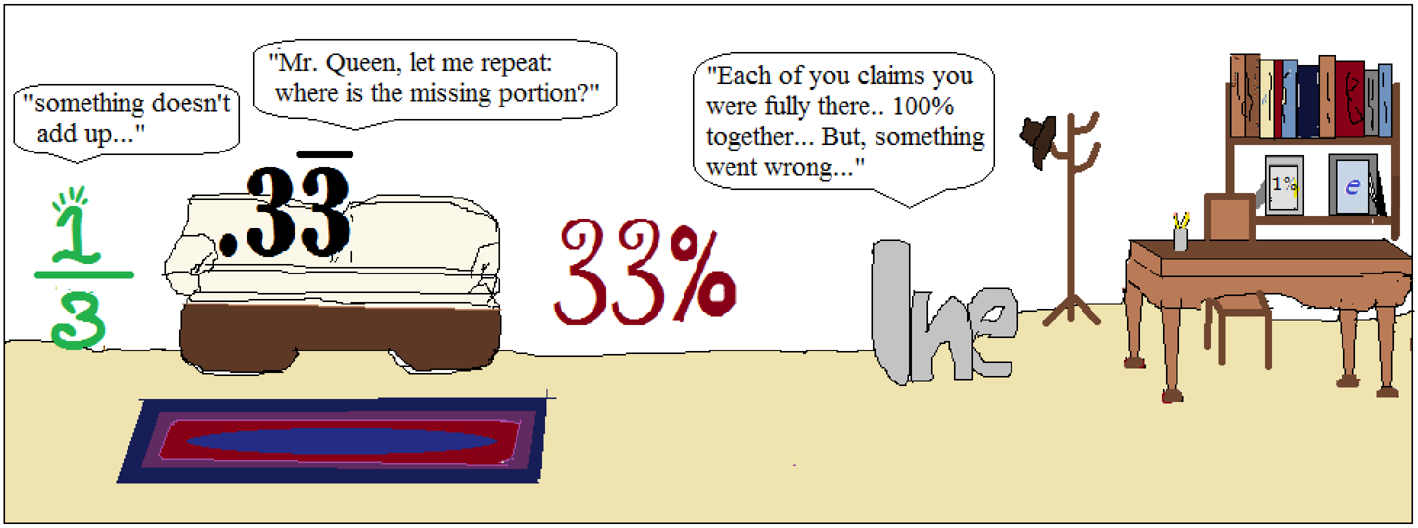
$$\prod_{i=2}^{n+1} \frac{i-1}{i} = \frac{1}{n} \cdot \frac{(n+1)-1}{n+1}$$

$$= \frac{1}{n} \cdot \frac{n}{n+1}$$

$$= \frac{1}{n+1}$$ ✓

(1975) Ine Queen and *The Case of the 3 Portions...*

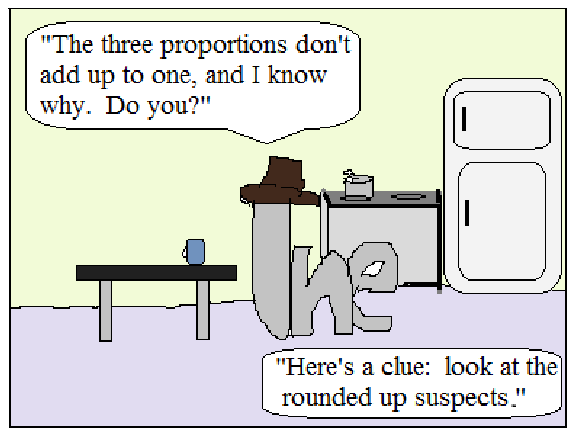
Ellery's Cousin



Ine Queen is the *one* detective for any math mystery (naturally)...

$.3\overline{3} + 33\% + \frac{1}{3} \neq 1$

What is the missing piece?



LanceAF #89 6-7-13
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Practice Exercises -→

Math Induction Practice

Use induction to prove the following:

1) $$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$$

2) $$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Use induction to prove the following:

Math Induction Practice

3) $(n-1)(n)(n+1)$ is divisible by 6 for all integers n

4) Show that the geometric series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ \ldots$ converges to 2.

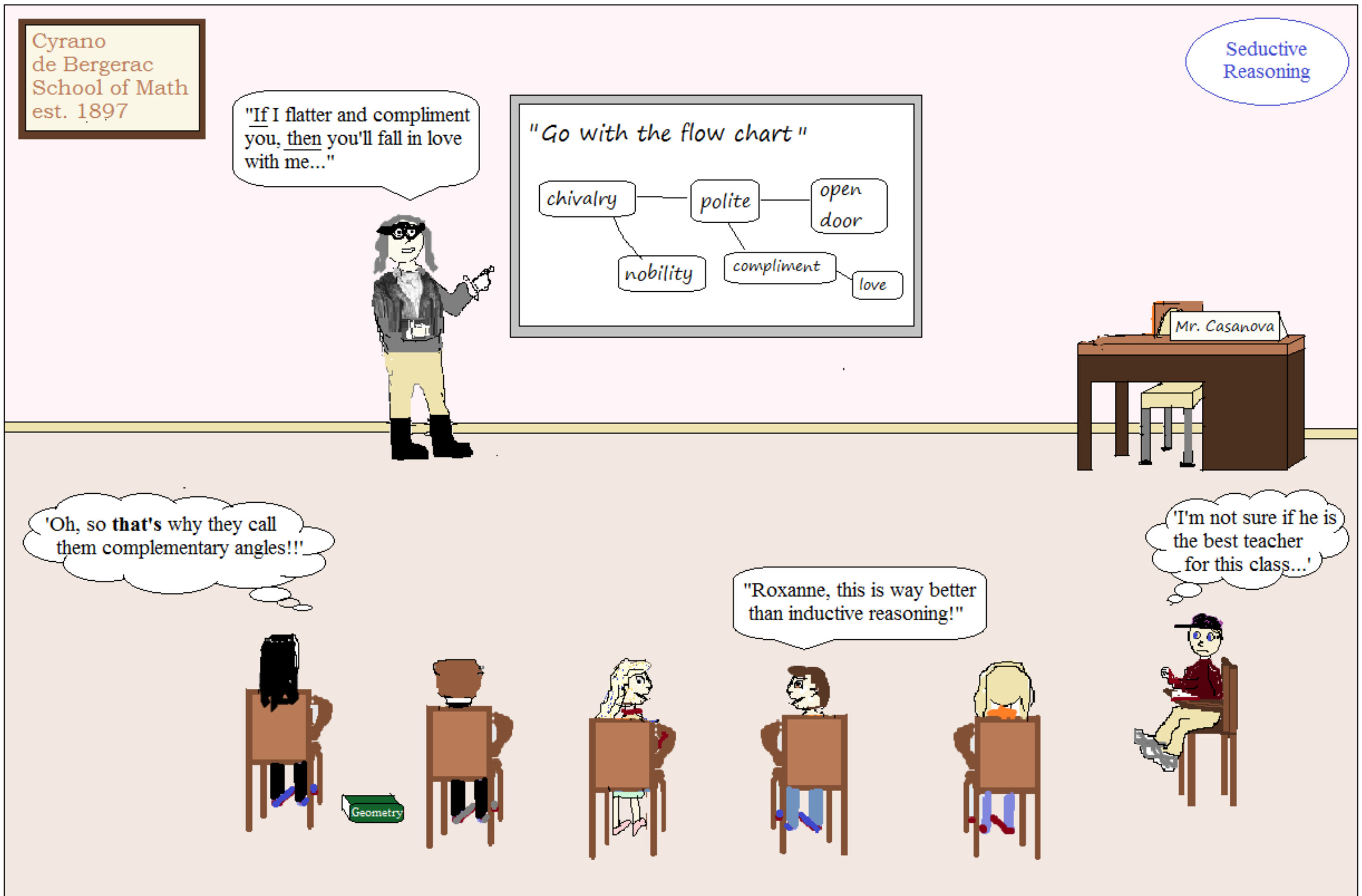
5) Prove: $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

6) Prove: $3 + 5 + 7 + \ldots + (2n + 1) = n^2 + 2n$

7) Prove $3^{2n} - 1$ is divisible by 8 for $n \geq 0$

8) Prove: $3n^2 + 3n$ is divisible by 6

9) Prove: $1 + 8 + 27 + 64 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$



LanceAF #337 (11-11-18)
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Using (un)conditional (love) statements,
Casanova teaches logic and reasoning to geometry students!

Solutions-→

Use induction to prove the following:

Math Induction Practice

SOLUTIONS

1) $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$

Step 1: Verify

If n = 1, then $1(1+1) = \frac{1(1+1)(1+2)}{3}$

$2 = 2$

If n = 4, then $2 + 6 + 12 + 20 = \frac{4(5)(6)}{3}$

$40 = 40$

So, we'll assume the formula is correct...

Step 2: Find n + 1, using the formula...

$$\sum_{k=1}^{n+1} k(k+1) = \frac{(n+1)(n+1+1)(n+1+2)}{3} = \frac{(n+1)(n+2)(n+3)}{3}$$

Step 3: Use confirmed formula (from step 1) and add the next term..

$$2 + 6 + 12 + 40 \ldots + n(n+1) + (n+1)(n+2) = \frac{n(n+1)(n+2)}{3} + (n+1)(n+2)$$

$$= \frac{n(n+1)(n+2)}{3} + \frac{3(n+1)(n+2)}{3}$$ GCF and regroup factors..

$$= \frac{(n+1)(n+2)(n+3)}{3}$$

Using successive term OR using the direct formula get same result.. ✓

2) $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

Step 1: Verify for n = 1

$(1)^3 = \frac{(1)^2(1+1)^2}{4} = 1$

also, if n = 2 $\quad (1)^3 + (2)^3 = \frac{(2)^2(2+1)^2}{4} = 9$

if n = 3 $\quad (1)^3 + (2)^3 + (3)^3 = \frac{(3)^2(3+1)^2}{4} = 36$

Step 2: Evaluate (using the formula) for n + 1

$$\sum_{k=1}^{n+1} k^3 = \frac{(n+1)^2((n+1)+1)^2}{4} = \frac{(n+1)^2(n+2)^2}{4}$$

Step 3: Since we know the formula worked for n = 1 (and n = 2 and 3), we'll assume it's correct. Now, let's add the next term....

$$1 + 9 + 36 + \ldots + n^3 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3$$

"split the term"

$$= \frac{n^2(n+1)^2}{4} + (n+1)^2(n+1)$$

common denominator

$$= \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^2(n+1)}{4}$$

condense

$$= \frac{n^2(n+1)^2 + (4n+4)(n+1)^2}{4}$$

regroup and

$$= \frac{(n+1)^2(n^2+4n+4)}{4}$$

factor

$$= \frac{(n+1)^2(n+2)^2}{4}$$

The result of step 3 matches the formula result in step 2! ✓

Use induction to prove the following: SOLUTIONS

Math Induction Practice

3) $(n-1)(n)(n+1)$ is divisible by 6 for all integers n

Step 1: Verify it works

if $n = 1$ $(1-1)(1)(1+1) = 0$

if $n = 2$ $(2-1)(2)(2+1) = 6$ each is divisible by 6

if $n = 3$ $(3-1)(3)(3+1) = 24$

Step 2: Assume the statement is true for k

$(k-1)(k)(k+1) = 6Z$ where Z is any integer (6Z must be divisible by 6)

$k^3 - k = 6Z$

Step 3: Prove for the next term $(k+1)$

$((k-1)+1)(k+1)((k+1)+1) =$

$(k)(k+1)(k+2) =$

$(k^2+k)(k+2) =$

$k^3 + 3k^2 + 2k =$

$k^3 + 3k^2 + 3k - k =$

$k^3 - k + 3k^2 + 3k =$

6Z

Then, is $3k^2 + 3k$ a multiple of 6?

if $3k^2 + 3k = 6Z$

then, $k^2 + k = 2Z$

So, is $k^2 + k$ always even?

$k(k+1)$ is always even!!

an even x odd number is even..

4) Show that the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ converges to 2.

Let $S = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n}\right)$

Then, $\frac{1}{2}S = \lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n \cdot 2}\right)$

Subtract: $S - \frac{1}{2}S = \lim_{n\to\infty}\left(1 - \frac{1}{2^{n+1}}\right)$

$\frac{1}{2}S = \lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{1}{2^{n+1}}$

$\frac{1}{2}S = 1 - 0$

$\frac{1}{2}S = 1$

$S = 2$

5) Prove: $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

SOLUTIONS

Math Induction Practice

Step 1: Try a few numbers to see if possible...

If $n = 1$, $\frac{1}{2!} = \frac{1}{2}$ $\quad 1 - \frac{1}{(1+1)!} = \frac{1}{2}$ ✓

If $n = 2$, $\frac{1}{2!} + \frac{2}{3!} = \frac{5}{6}$ $\quad 1 - \frac{1}{(2+1)!} = \frac{5}{6}$ ✓

Step 2: Assume for all integers k

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$$

Step 3: Confirm, if works for next term...

Prove: $$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+2)!}$$

$$1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!}$$

$$1 - \frac{1}{(k+1)!} \cdot \frac{(k+2)}{(k+2)} + \frac{k+1}{(k+2)!}$$

$$1 - \frac{(k+2)}{(k+2)!} + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+2)!}$$ ✓

6) Prove: $3 + 5 + 7 + \dots + (2n+1) = n^2 + 2n$

Step 1: Test assumption

$n = 1 \quad 2(1) + 1 = 3 = (1)^2 + 2(1)$ ✓

$n = 2 \quad 3 + 5 = 8 = (2)^2 + 2(2)$ ✓

$n = 3 \quad 3 + 5 + 7 = 15 = (3)^2 + 2(3)$ ✓

Step 2: Assume for all integers k

$$3 + 5 + 7 + \dots + (2k+1) = k^2 + 2k$$

Step 3: Prove for the next term....

$$3 + 5 + 7 + \dots + (2k+1) + (2(k+1) + 1) = (k+1)^2 + 2(k+1)$$

$$k^2 + 2k \quad + 2k + 3$$

(split the 3)

$$k^2 + 2k + 1 \quad + 2k + 2$$

$$(k+1)^2 \quad + 2(k+1) \quad = (k+1)^2 + 2(k+1)$$ ✓

Mathematical Induction Practice

SOLUTIONS

7) Prove $3^{2n} - 1$ is divisible by 8 for $n \geq 0$

If $n = 0$, $3^0 - 1 = 0$

If $n = 1$, $3^2 - 1 = 8$

If $n = 2$, $3^4 - 1 = 80$

$3^{2n} - 1 = 8Z \Rightarrow 3^{2n} = 8Z + 1$

$3^{2(n+1)} - 1 = 3^{2n+2} - 1$

$= 3^{2n} \cdot 3^2 - 1$

substitute

$9 \cdot 3^{2n} - 1 \Rightarrow 9 \cdot (8Z + 1) - 1 = 72Z + 9 - 1 = 72Z + 8$

$8(9Z + 1)$ is divisible by 8!!! ✓

8) Prove: $3n^2 + 3n$ is divisible by 6

Let n = 1 ----> 6 ✓

Let n = 2 ----> 18 ✓

Let n = 3 ----> 36 ✓

General assumption: $3k^2 + 3k = 6P$

Prove using induction: next term....

$3(k+1)^2 + 3(k+1) =$

$3k^2 + 6k + 3 + 3k + 3 =$

rearrange

$3k^2 + 3k + 6k + 3 + 3 =$

$6P + 6k + 6$

$6P + 6(k+1)$

6P is divisible by 6 (by assumption)
and
6(k + 1) must be divisible by 6! ✓

9) Prove: $1 + 8 + 27 + 64 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

Test: If $n = 2$, $1 + 8 = 9$ $\left(\frac{2(2+1)}{2}\right)^2 = 9$ ✓

If $n = 3$, $1 + 8 + 27 = 36$ $\left(\frac{3(3+1)}{2}\right)^2 = 36$ ✓

General: $1 + 8 + 27 + 64 + ... + k^3 = \left(\frac{k(k+1)}{2}\right)^2$

Induction by adding next term: $1 + 8 + 27 + 64 + ... + k^3 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$

$\left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$

$\frac{k^2(k+1)^2}{2^2} + (k+1)^2(k+1)$

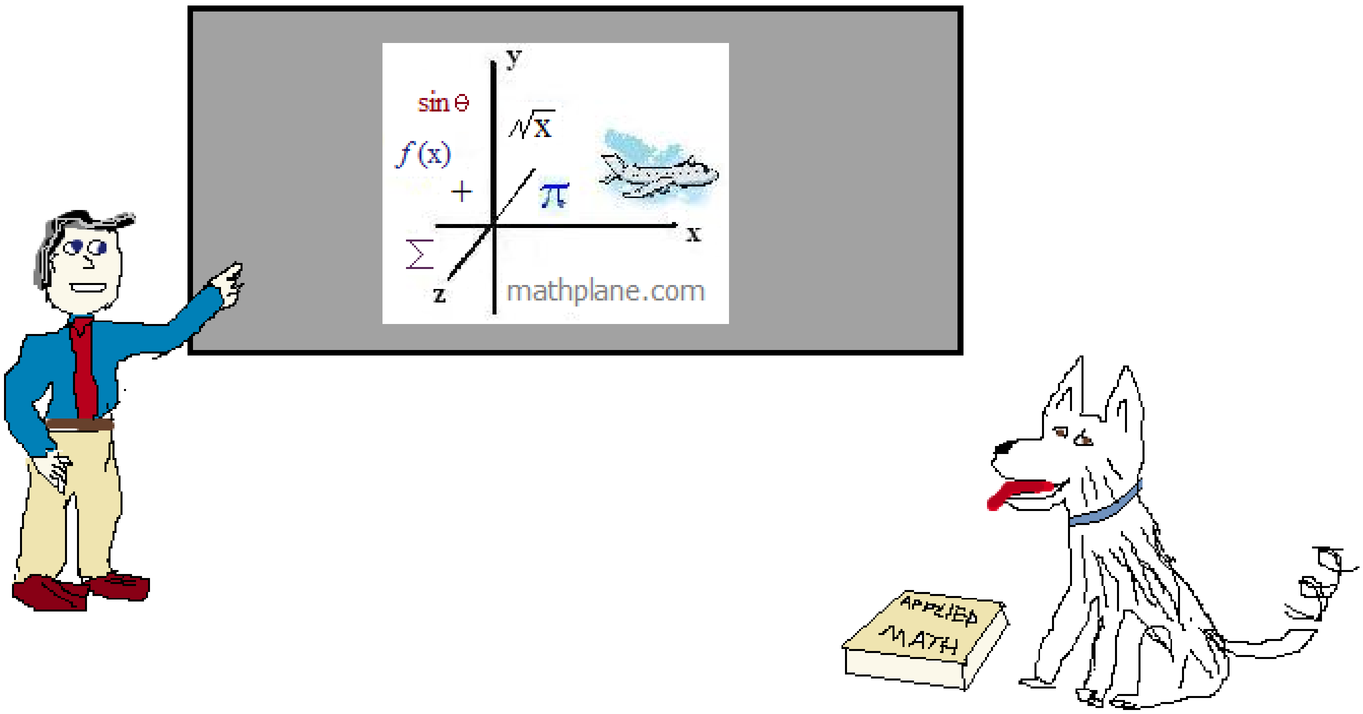
$\frac{k^2(k+1)^2}{2^2} + \frac{4(k+1)^2(k+1)}{2^2}$

$(k+1)^2 \cdot \left[\frac{k^2 + 4k + 4}{2^2}\right] = (k+1)^2\left[\frac{(k+2)(k+2)}{2^2}\right] = \left(\frac{(k+1)(k+2)}{2}\right)^2$ ✓

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Thanks.



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