Calculus: Chain Rule

Notes, Examples, and Practice Quiz (with Solutions)

(composite function)

If $h(x) = f(g(x))$

then $h'(x) = f'(g(x)) \cdot g'(x)$

If $y = f(u)$ and $u = g(x)$

then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

Topics include related rates of change, conversions, composite functions, derivatives, power rule, and more.

Mathplane.com

Chain Rule

Derivatives show the rates of change between variables.

Example: $y = 2x^2 + 4x + 7$

$\frac{dy}{dx} = 4x + 4$ The instantaneous rate of change of y with respect to x is $4x + 4$.

But, what happens when other rates of change are introduced?

Chain Rule Equivalents:

Example: A car's velocity is 60 miles per hour. How many miles per minute does it travel?

Its rate of change (derivative with respect to hours) is $\frac{60 \text{ miles}}{1 \text{ hour}}$

But, its (velocity) rate of change with respect to minutes

$$\frac{60 \text{ miles}}{1 \text{ hour}} \text{ X } \frac{1 \text{ hour}}{60 \text{ minutes}} = 1 \frac{\text{mile}}{\text{minute}}$$

Notice: We needed to introduce another ratio to get our desired rate of change!

Example: Derivative is rate of change -- price per unit..

$\frac{\triangle \text{Quantity}}{\triangle \text{Price \$}}$ candy bar price in $US $\frac{\text{candy bar}}{\$}$

But, what if you wanted to find the price of the candy bar in pesos?

In other words, you want to find the rate of change of candy bars with respect to pesos

$$\frac{\text{candy bar}}{\$} \text{ X } \frac{1 \text{ \$}}{12 \text{ Pesos}} = \frac{\text{candy bar}}{12 \text{ pesos}}$$

$$\frac{dQ}{d\$} \qquad \frac{d\$}{dP} \qquad \frac{dQ}{dP}$$

Similarly, composite functions can have related rates:

$$\frac{dy}{du} \text{ X } \frac{du}{dx} = \frac{dy}{dx}$$

Chain Rule

Composite Functions. consist of multiple functions ---> multiple rates of change and relationships between inputs/outputs

Each of the following can be expressed as 2 or more functions:

$h(x) = \sqrt{2x + 1}$ $\quad f(x) = \sqrt{x}$ $\quad g(x) = 2x + 1$

$h(x) = (3x + 9)^5$ $\quad f(x) = x^5$ $\quad g(x) + 3x + 9$

$h(x) = \sin^4 x$ $\quad f(x) = x^4$ $\quad g(x) = \sin x$

$p(t) = \cos^2(3t + 5)$ $\quad f(t) = t^2$ $\quad g(t) = \cos(t)$ $\quad h(t) = 3t + 5$

To find the derivative of a composite function:

Chain Rule Formula

if $h(x) = f(g(x))$ (composite function)

then $h'(x) = f'(g(x)) \cdot g'(x)$

if $y = f(u)$ and $u = g(x)$ (separated function/substitution)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example Find the derivative of $h(x) = (x^3 + 3x^2 + 6)^4$

(One approach is to expand the entire function. Then, take the derivative. But, that would be time consuming)

We'll use the power and chain rules to find $h'(x)$: viewing $h(x)$ as a composite function $f(g(x))$

$f(x) = x^4$ $\quad g(x) = x^3 + 3x^2 + 6$

$f'(x) = 4x^3$

$g'(x) = 3x^2 + 6x + 0$

$h'(x) = \underbrace{4(x^3 + 3x^2 + 6)^3}_{f'(g(x))} \cdot \underbrace{(3x^2 + 6x)}_{g'(x)}$

Shortcut: $h(x) = (x^3 + 3x^2 + 6)^4$

$h'(x) = (x^3 + 3x^2 + 6)^4$ — exponent in front of parenthesis

$4(x^3 + 3x^2 + 6)$

$4(x^3 + 3x^2 + 6)^3$ ← new exponent: 'minus 1'

$4(x^3 + 3x^2 + 6)^3 \cdot (3x^2 + 6x)$ — multiply by derivative of terms inside the parenthesis

Using the product rule to verify the chain rule

Example: Find the derivatives of $(x^3+7)^2$, $(x^3+7)^3$, and $(x^3+7)^4$ to derive $(x^3+7)^n$

using product rule, the derivative of $(x^3+7)^2$

$$y = (x^3+7)(x^3+7)$$

$$y' = (3x^2+0)(x^3+7) + (x^3+7)(3x^2+0)$$

$$= 2\cdot(x^3+7)(3x^2+0)$$

$$= 2\cdot(x^3+7)(3x^2)$$

and, the derivative of $(x^3+7)^3$

$$y = (x^3+7)^2(x^3+7)$$

$$y' = \underbrace{2\cdot(x^3+7)(3x^2)}_{\text{from above}}(x^3+7) + (3x^2+0)(x^3+7)^2$$

$$= 2\cdot(x^3+7)^2(3x^2) + (3x^2)(x^3+7)^2$$

$$= 2(x^3+7)^2(3x^2) + (x^3+7)^2(3x^2)$$

$$= 3(x^3+7)^2(3x^2)$$

and, the derivative of $(x^3+7)^4$

$$y = \underset{f}{(x^3+7)^3}\ \underset{g}{(x^3+7)}$$

$$y' = \underbrace{\overset{f'}{3(x^3+7)^2(3x^2)}}_{\text{from above}}\cdot\overset{g}{(x^3+7)} + \overset{g'}{(3x^2+0)}\overset{f}{(x^3+7)^3}$$

$$= 3(x^3+7)^3(3x^2) + (3x^2)(x^3+7)^3$$

$$= 4(x^3+7)^3(3x^2)$$

We see a pattern: $y = (x^3+7)^n$ $\qquad y' = n(x^3+7)^{n-1}(3x^2)$

derivative

Power and Chain Rule -- 3 approaches

Example: $y = (x^2 + 3x + 1)^3$

Separate the composite function

$f(x) = x^3 \quad f'(x) = 3x^2$

$g(x) = (x^2 + 3x + 1)$

... find derivatives of each part...

$g'(x) = 2x + 3$

Apply the formula

$y' = 3(x^2 + 3x + 1)^2 (2x + 3)$

$f'(g(x)) \qquad g'(x)$

Chain Rule Formula

if $h(x) = f(g(x))$ (composite function)

then $h'(x) = f'(g(x)) \cdot g'(x)$

Example. find the derivative of $(3x^3 + 2x - 7)^4$

exponent in front: $(3x^3 + 2x - 7)^4$

$4(3x^3 + 2x - 7)$

exponent - 1

$4(3x^3 + 2x - 7)^3$

times derivative of parenthesis

$4(3x^3 + 2x - 7)^3 \cdot (9x^2 + 2 - 0)$

Example: Use substitution to find the derivative of $y = (5x^2 + 3x + 4)^3$

Substitute U variable

Let $U = 5x^2 + 3x + 4$

so, $y = U^3$

if $y = f(u)$ and $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Find individual rates of change

$$\frac{dU}{dx} = 10x + 3$$

$$\frac{dy}{dU} = 3U^2$$

Apply chain rule to find dy/dx

$$\frac{dy}{dx} = \frac{dy}{dU} \cdot \frac{dU}{dx} \longrightarrow 3U^2 \cdot (10x + 3)$$

basic fractions!

Substitution: $3(5x^2 + 3x + 4)^2 (10x + 3)$

Example: For the equation $x = \sin t + 3$, $y = \cos t$ find $\frac{dy}{dx}$

We can find $\frac{dx}{dt} \longrightarrow \cos t + 0$

then, find $\frac{dy}{dt} \longrightarrow -\sin t$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin t}{\cos t} = \boxed{-\tan t}$$

These show how x and y change *as t changes*

To find $\frac{dy}{dx}$, we combine the fractions (rates of change)

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dx}$$

Example: Find the line tangent to $x = \cos t$, $y = 1 - \sin t$ at $t = \frac{\pi}{6}$

substitute t into each equation to get the (x, y) coordinate

$$x = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$y = 1 - \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

Then, to find the slope, we need $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{0 - \cos t}{-\sin t}$ or $\cot(t)$

slope or $\frac{dy}{dx}$ at $t = \frac{\pi}{6}$

$\sqrt{3}$

at $t = \frac{\pi}{6}$ $\quad \frac{dy}{dx} = \sqrt{3}$

Then, write equation of the line: $y - \frac{1}{2} = \sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right)$ or, $\boxed{y = \sqrt{3}\,x - 1}$

Example: $y = 3x^2 + 6x - 10$

If $\frac{dx}{dt} = 4t$ and $\frac{dy}{dt} = -3t$ where is the point at t = 3 ?

rates of change:

$\frac{dy}{dx} = 6x + 6$

$\frac{dx}{dt} = 4t$

$\frac{dy}{dt} = -3t$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = 6x + 6 \quad \frac{dy}{dx}$$

at t = 3, dy/dt = -9

dx/dt = 12

Therefore, the instantaneous rate of change

$6x + 6 = -3/4$

$6x = -9/2$

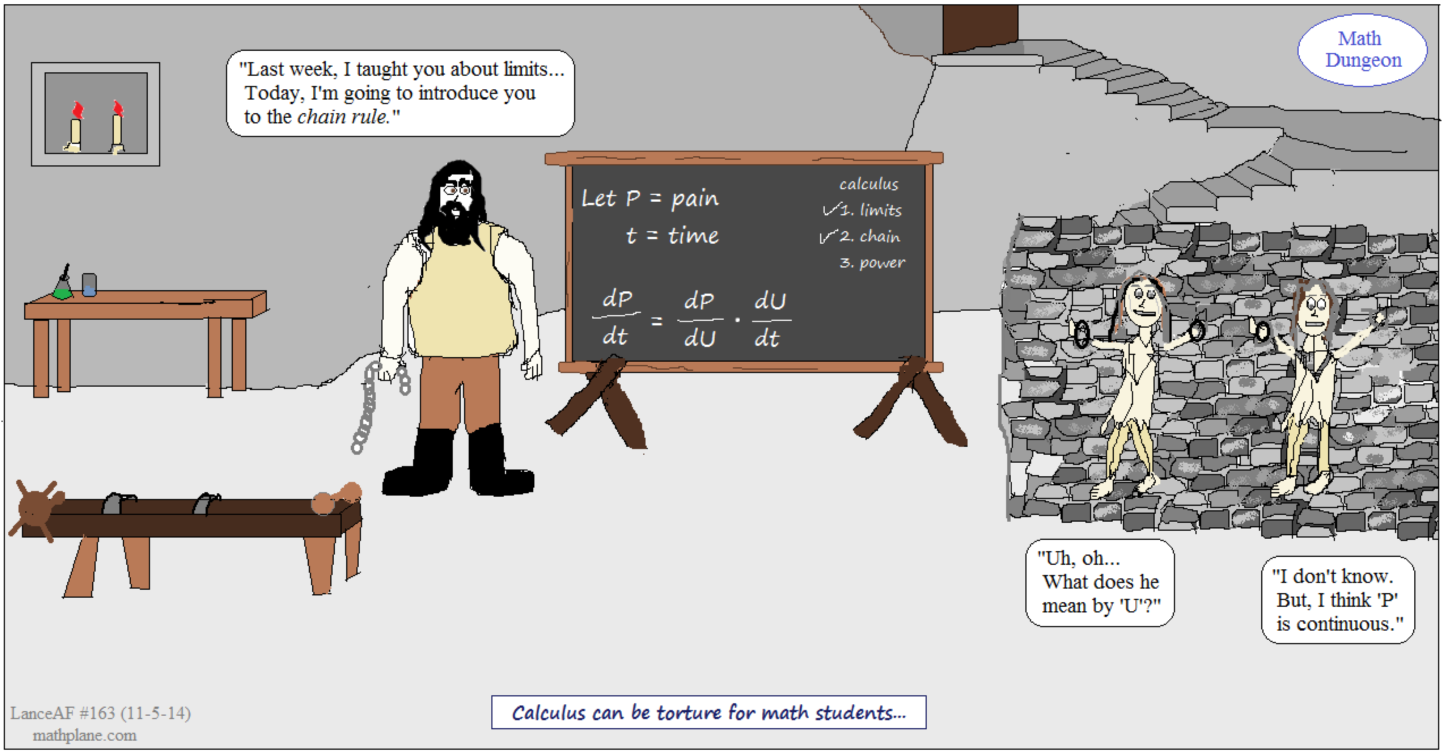
$x = -3/4$

at t = 3, x will be at -3/4...

and, y will be at -205/16

$y = 3(-3/4)^2 + 6(-3/4) - 10 = -205/16$

(-3/4, -205/16)



Practice Quiz-→

Chain Rule Quiz

1) Separate the following composite functions:

a) $h(x) = \sqrt{5x + 1}$

b) $h(x) = (8x + 9)^3$

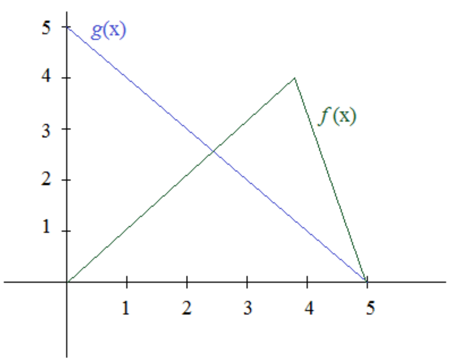
c) $h(x) = \sin^2 x$

d) $p(t) = \cos^3 (3t - 5)$

e) $f(t) = 2e^{(3t + 7)}$

2) Evaluate the following derivatives, using the graph below.

Note:
If $h(x) = f(g(x))$, then $h'(x) = f'(g(x)) \cdot g'(x)$



$h'(3) =$

$h'(2) =$

$h'(1) =$

$h'(1/2) =$

3) Find the derivative of $y = (x^3 + 3)^2$ using the following appoaches:

a) Multiply/expand the equation

b) Product rule

c) Use chain rule

Chain Rule Quiz

4) find the tangent line at $t = 3$

$x = 3t^2 + 1$

$y = -2t$

5) Find the first and second derivatives:

a) $(4x^6 + 3x + 7)^2$

b) $\dfrac{1}{\sqrt{x^3 + 5}}$

6) Find the (first) derivatives of the following:

a) $(x - 1)^3 \cdot (7x + 4)^2$

b) $(1 + \cos^2 7x)^3$

7) Using the tables, find the following,

Chain Rule Quiz

A) D_x of $g(f(x))$ when x = 2

x	1	2	3	4
$f(x)$	3	1	2	4
$f'(x)$	-7	-5	-4	-6
$g(x)$	4	3	1	2
$g'(x)$	1/3	1/9	7/9	2/9

a) 2/9

b) -4

c) 7/9

d) -5/3

e) -28/9

B) D_x of $g(f(x))$ when x = 1

x	1	2	3	4
$f(x)$	3	1	2	4
$f'(x)$	-2	-5	-4	-6
$g(x)$	4	3	1	2
$g'(x)$	1/3	1/9	7/9	2/9

C) D_x of $f(g(x))$ when x = 3

x	1	2	3	4
$f(x)$	3	1	2	4
$f'(x)$	-7	-5	-4	-6
$g(x)$	4	3	1	2
$g'(x)$	1/3	1/9	7/9	2/9

Solutions-→

Chain Rule Quiz

SOLUTIONS

1) Separate the following composite functions:

a) $h(x) = \sqrt{5x+1}$ $\quad f(x) = \sqrt{x}$ $\quad g(x) = 5x + 1$

b) $h(x) = (8x+9)^3$ $\quad f(x) = x^3$ $\quad g(x) + 8x + 9$

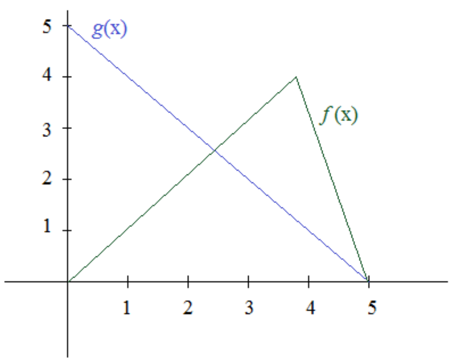
c) $h(x) = \sin^2 x$ $\quad f(x) = x^2$ $\quad g(x) = \sin x$

d) $p(t) = \cos^3(3t - 5)$ $\quad f(t) = t^3$ $\quad g(t) = \cos(t)$ $\quad h(t) = 3t - 5$

e) $f(t) = 2e^{(3t+7)}$ $\quad f(t) = 2e^t$ $\quad g(t) = 3t + 7$

2) Evaluate the following derivatives, using the graph below.

Note:
If $h(x) = f(g(x))$, then $h'(x) = f'(g(x)) \cdot g'(x)$



$h'(3) =$ $\quad g(3) = 2$ $\quad f'(2) = 1$ (slope $f(x)$ at $x = 2$)

$g'(3) = -1$ (slope of $g(x)$ at $x = 3$)

$h'(3) = f'(g(3)) \cdot g'(3) = (1)\ (-1) = \boxed{-1}$

$h'(2) =$ $\quad g(2) = 3$ $\quad f'(3) = 1$

$g'(2) = -1$ $\quad h'(2) = f'(g(2)) \cdot g'(2) = \boxed{-1}$

$h'(1) =$ $\quad g(1) = 4$ $\quad f'(4) =$ undefined (it's a corner)

$g'(1) = -1$ $\quad h'(1) = \boxed{\text{undefined}}$

$h'(1/2) =$ $\quad g(1/2) = 4.5$ $\quad f'(4.5) = -4$ (slope of line between 4 and 5)

$g'(1/2) = -1$ $\quad h'(1/2) = f'(g(1/2)) \cdot g'(1/2) = -4 \cdot -1 = \boxed{4}$

3) Find the derivative of $y = (x^3 + 3)^2$ using the following appoaches:

a) Multiply/expand the equation

$y = (x^3 + 3)(x^3 + 3)$

$y = x^6 + 6x^3 + 9$

$y' = 6x^5 + 18x^2$

b) Product rule

$y = \underset{f}{(x^3 + 3)}\underset{g}{(x^3 + 3)}$

$y' = \underset{f'}{(3x^2 + 0)}\underset{g}{(x^3 + 3)} + \underset{g'}{(3x^2 + 0)}\underset{f}{(x^3 + 3)}$

$y' = 3x^5 + 9x^2 + 3x^5 + 9x^2$

$y' = 6x^5 + 18x^2$

c) Use chain rule

$y = (x^3 + 3)^2$

$y = U^2$ $\qquad U = (x^3 + 3)$

$\frac{dy}{dU} = 2U$ $\qquad \frac{dU}{dx} = 3x^2 + 0$

$\frac{dy}{dx} = \frac{dy}{dU} \cdot \frac{dU}{dx}$

$= 2U \cdot 3x^2$

$= 2(x^3 + 3) \cdot 3x^2$

$= 6x^5 + 18x^2$

SOLUTION

Chain Rule Quiz

4) find the tangent line at $t = 3$

$x = 3t^2 + 1$

$y = -2t$

at $t = 3$, $x = 13$ (13, -6)

$y = -6$

$\frac{dx}{dt} = 6t$

$\frac{dy}{dt} = -2$

at $t = 3$,

$\frac{dx}{dt} = 18$

$\frac{dy}{dt} = -2$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2}{18}$$

slope is -1/9

Tangent line:

$y + 6 = -1/9(x - 13)$

5) Find the first and second derivatives:

a) $(4x^6 + 3x + 7)^2$

first derivative: $2(4x^6 + 3x + 7)^1 \cdot (24x^5 + 3)$

$(8x^6 + 6x + 14) \cdot (24x^5 + 3)$

second derivative: product rule

$(48x^5 + 6 + 0)(24x^5 + 3) + (120x^4 + 0)(8x^6 + 6x + 14)$

f' g g' f

$(48x^5 + 6)(24x^5 + 3) + (120x^4)(8x^6 + 6x + 14)$

b) $\frac{1}{\sqrt{x^3 + 5}}$

rewrite: $(x^3 + 5)^{\frac{-1}{2}}$

first derivative: $\frac{-1}{2}(x^3 + 5)^{\frac{-3}{2}} \cdot (3x^2 + 0)$

$$\frac{-3x^2}{2\sqrt{(x^3 + 5)^3}}$$

second derivative:

$\frac{-1}{2}(x^3 + 5)^{\frac{-3}{2}} \cdot (3x^2)$

product rule: $\frac{-3x^2}{2} \cdot (x^3 + 5)^{\frac{-3}{2}}$

$-3x \cdot (x^3 + 5)^{\frac{-3}{2}} + \frac{-3}{2}(x^3 + 5)^{\frac{-5}{2}}(3x^2) \cdot \frac{-3x^2}{2}$

f' g g' f

$$\frac{-3x}{\sqrt{(x^3 + 5)^3}} + \frac{27x^4}{4\sqrt{(x^3 + 5)}}$$

6) Find the (first) derivatives of the following:

a) $(x - 1)^3 \cdot (7x + 4)^2$

first derivative:

product rule and chain rule...

$3(x - 1)^2 \cdot (7x + 4)^2 + 2(7x + 4)(7) \cdot (x - 1)^3$

GCF and simplify...

$(x - 1)^2(7x + 4)\left[3(7x + 4) + 14(x - 1)\right]$

b) $(1 + \cos^2 7x)^3$

$y' = 3(1 + \cos^2 7x)^2 \cdot [0 + 2(\cos 7x)^1(-\sin 7x)(7)]$

$= 3(1 + \cos^2 7x)^2 [-14(\cos 7x)(\sin 7x)]$

$= -42(1 + \cos^2 7x)^2 (\cos 7x)(\sin 7x)$

7) Using the tables, find the following,

Chain Rule Quiz

SOLUTIONS

A) D_x of $g(f(x))$ when x = 2

x	1	2	3	4
$f(x)$	3	1	2	4
$f'(x)$	-7	-5	-4	-6
$g(x)$	4	3	1	2
$g'(x)$	1/3	1/9	7/9	2/9

a) 2/9

b) -4

c) 7/9

d) -5/3

e) -28/9

Using the chain rule of compositions,
Derivative: $g(f(x))' = g'(f(x)) \cdot f'(x)$

$f(2) = 1$

$g'(1) = 1/3$

$f'(2) = -5$

-5/3

B) D_x of $g(f(x))$ when x = 1

x	1	2	3	4
$f(x)$	3	1	2	4
$f'(x)$	-2	-5	-4	-6
$g(x)$	4	3	1	2
$g'(x)$	1/3	1/9	7/9	2/9

Using the chain rule of compositions,
Derivative: $g(f(x))' = g'(f(x)) \cdot f'(x)$

$f(1) = 3$

$g'(3) = 7/9$ $\quad 7/9 \cdot -2 =$ -14/9

$f'(1) = -2$

C) D_x of $f(g(x))$ when x = 3

x	1	2	3	4
$f(x)$	3	1	2	4
$f'(x)$	-7	-5	-4	-6
$g(x)$	4	3	1	2
$g'(x)$	1/3	1/9	7/9	2/9

$f(g(x))' = f'(g(x)) \cdot g'(x)$

$g(3) = 1$

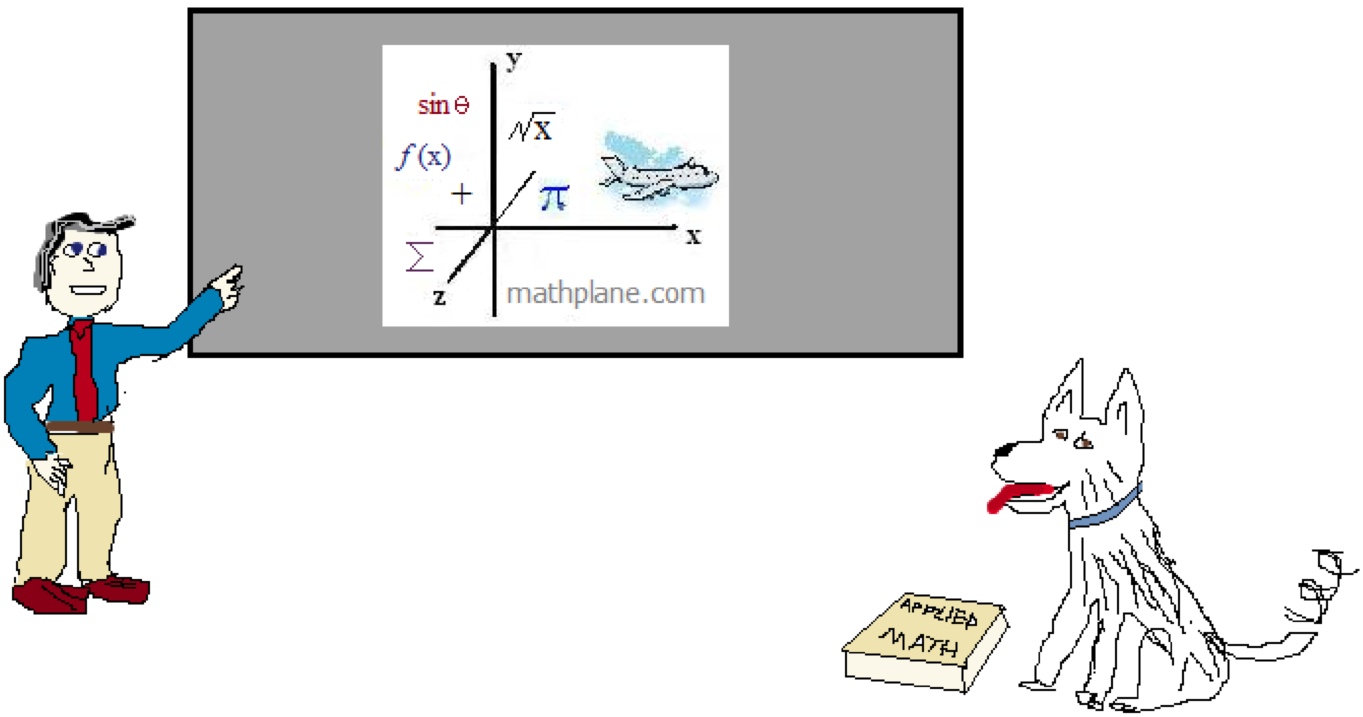
$f'(1) = -7$ $\quad -7 \cdot 7/9 =$ -49/9

$g'(3) = 7/9$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers



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