

Binomial Expansion

Notes, Examples, Formulas, and Practice

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{(n-k)} b^k$$

Topics include factorials, combinations, polynomial multiplication, Pascal's Triangle, and more

Binomial (Expansion) Theorem

where

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{(n-k)} b^k \quad \binom{n}{k} = \frac{n!}{(n-k)! k!} = {}_n C_k$$

Example: Expand the binomial $(2x + y)^4$

"Place the first terms" $(2x)^4 + (2x)^3 + (2x)^2 + (2x)^1 + (2x)^0$

"Place the second terms" $(2x)^4(y)^0 + (2x)^3(y)^1 + (2x)^2(y)^2 + (2x)^1(y)^3 + (2x)^0(y)^4$

"Add the coefficients" $\binom{4}{0}(2x)^4(y)^0 + \binom{4}{1}(2x)^3(y)^1 + \binom{4}{2}(2x)^2(y)^2 + \binom{4}{3}(2x)^1(y)^3 + \binom{4}{4}(2x)^0(y)^4$

Simplify $(1)(16x^4)(1) + (4)(8x^3)(y) + (6)(4x^2)(y^2) + (4)(2x)(y^3) + (1)(1)(y^4)$

$$16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$$

Finding the term:

where $n \geq (r - 1)$

The r^{th} term of the expansion $(a + b)^n$ is

$$\binom{n}{r-1} a^{n-(r-1)} b^{(r-1)}$$

Example: Find the 15th term of $(x^2 + y)^{22}$

$$\binom{22}{15-1} (x^2)^{22-(15-1)} (y)^{(15-1)}$$

$$\binom{22}{14} (x^2)^8 (y)^{14}$$

$$319,770x^{16}y^{14}$$

Alternative method (if you forget the formula)...

"Begin the binomial expansion and determine the pattern!"

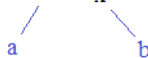
1st term	2nd term	3rd term
$\binom{22}{0} (x^2)^{22} (y)^0$	$+ \binom{22}{1} (x^2)^{21} (y)^1$	$+ \binom{22}{2} (x^2)^{20} (y)^2$

so, the 15th term will be... $\binom{22}{14}$

(observe: the difference between the 1st term and 15th term is 14 terms...)

$$(x^2)^8 (y)^{14}$$

Example: Find the constant term in the binomial expansion of $(x^2 + \frac{3}{x})^{15}$



since the 'a' and 'b' terms are multiplied, we need to figure out which term would cancel the variable x...
 In other words, which term will have an exponent in the 3/x term that is double the exponent in the x² term

$${}^{15}C_5 \cdot (x^2)^5 \left(\frac{3}{x}\right)^{10}$$

$$3003 \cdot x^{10} \cdot \frac{59049}{x^{10}} = 177324147$$

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

Note: each number is the sum of the adjoining number(s) above...

$\binom{6}{2}$ \Rightarrow 7th row, 3rd position... 15

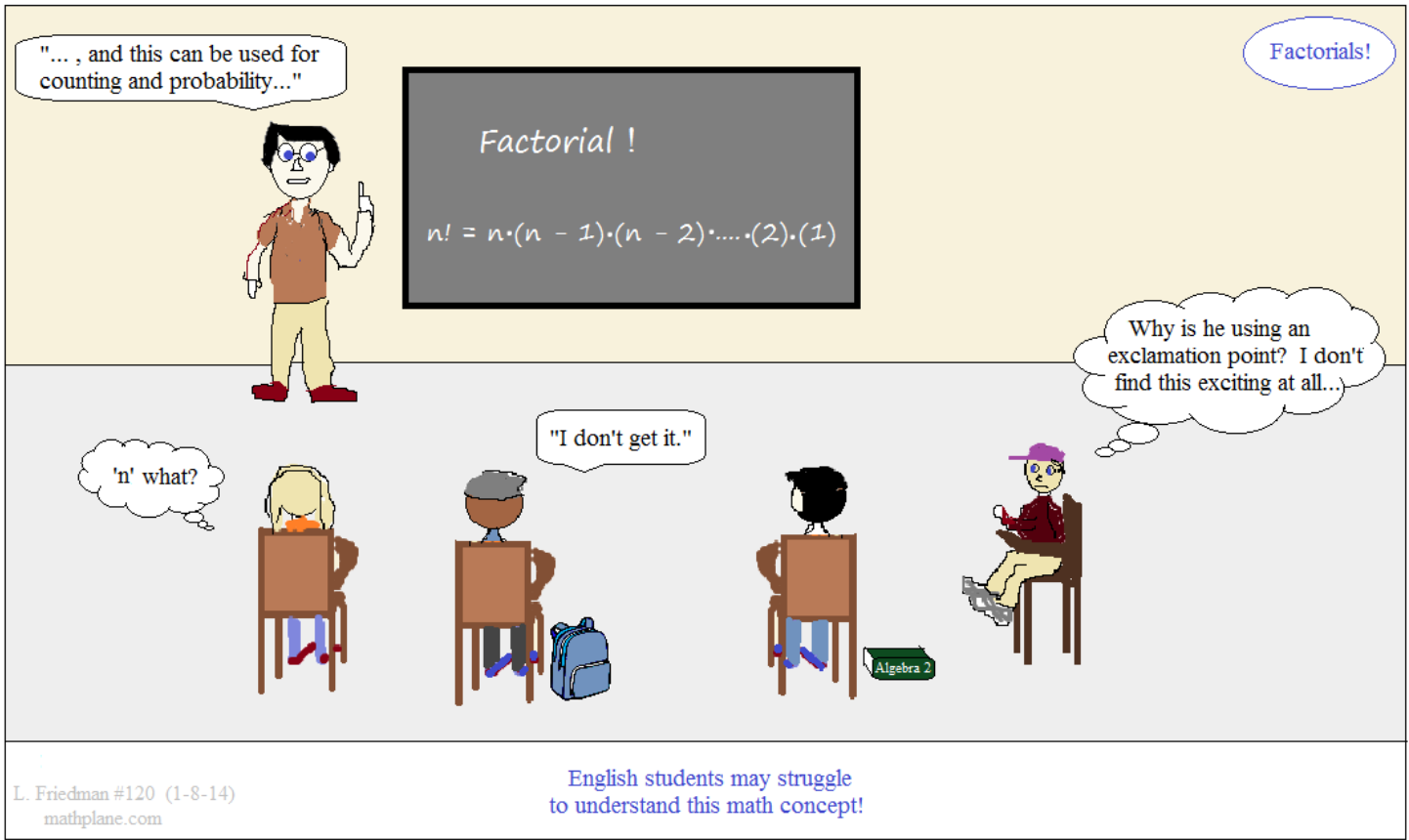
$$\frac{6!}{4! 2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{(4 \times 3 \times 2 \times 1) \times 2 \times 1} = \frac{30}{2} = 15$$

Example: What is $\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}$?

This is the 6th row in Pascal's Triangle...

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

1 + 5 + 10 + 10 + 5 + 1 = 32



Practice Exercises ->

I. Expand the following:

1) $(x + y)^5 =$

2) $(m - p)^7 =$

3) $(b + 2)^4 =$

4) $(2x - 3)^5 =$

5) $(x^2 + y^3)^6 =$

II. Find the term:

1) $(x + 4y)^7$ 5th term

2) $(3x - 2)^{12}$ 6th term

3) $(2m + p^2)^{24}$ 9th term

III. More Questions and Concepts

A) Condense/Simplify the following Binomial Expansions...

$$1) \binom{5}{0} a^{10} + \binom{5}{1} 3a^8 b + \binom{5}{2} 9a^6 b^2 + \binom{5}{3} 27a^4 b^3 + \binom{5}{4} 81a^2 b^4 + \binom{5}{5} 243b^5$$

$$2) \binom{4}{0} 5^4 + \binom{4}{1} 5^3 (-2)^1 + \binom{4}{2} 5^2 (-2)^2 + \binom{4}{3} 5^1 (-2)^3 + \binom{4}{4} (-2)^4$$

B) Use binomial expansion to find:

$$1) (1.001)^4 \quad \text{Hint: } .001 = 10^{-3}$$

$$2) (.998)^3 \quad \text{Hint: } .002 = 2(10)^{-3}$$

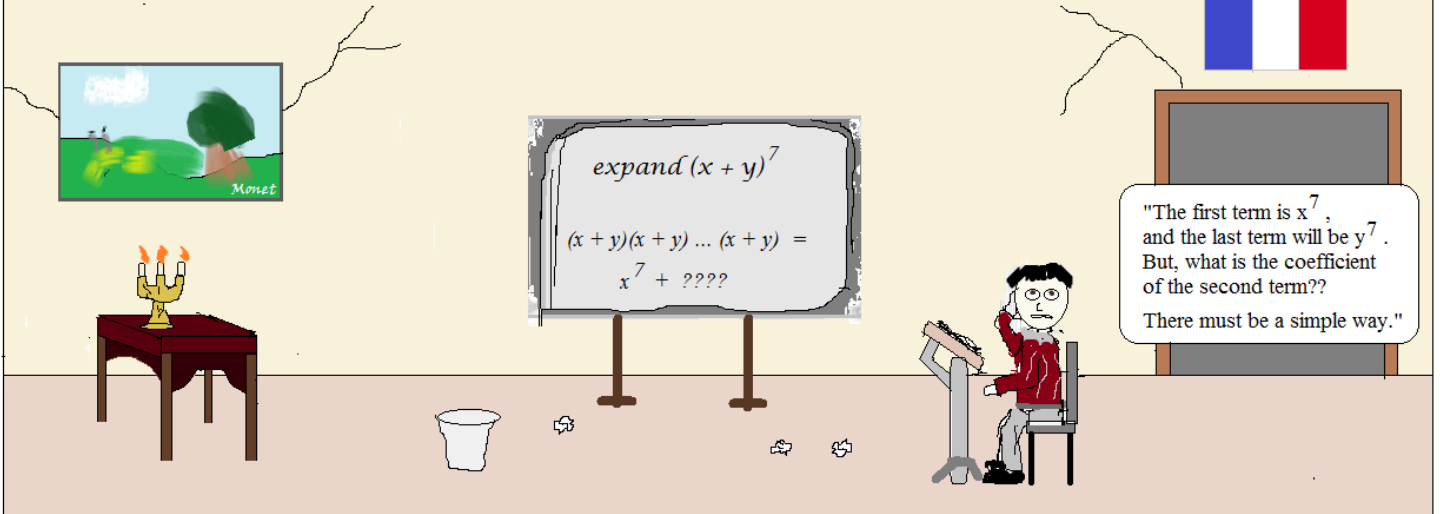
C) Miscellaneous

$$1) \text{ What is the 5th term in the expansion of } (x + 3y^2)^5 ?$$

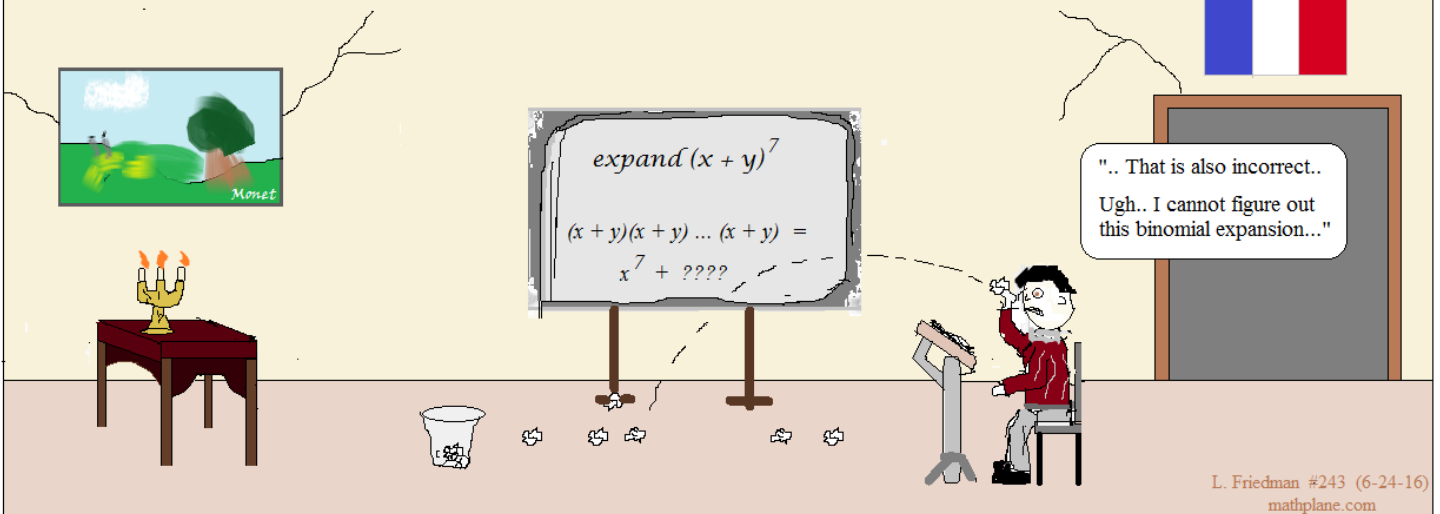
$$2) \text{ What is the coefficient of the } st^2 \text{ term in the expansion of } (s - 5t)^3 ?$$

$$3) \text{ In the expansion of } (2k + 2)^{18}, \text{ what is the term that includes } k^7 ?$$

One afternoon,

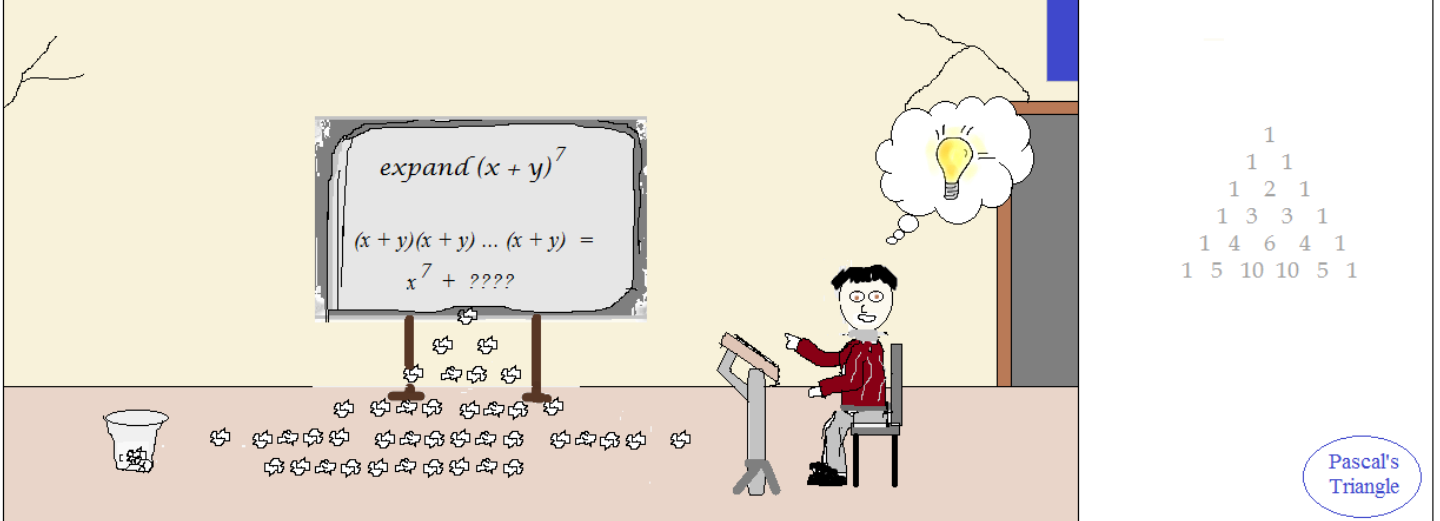


while struggling with his pre-calculus homework,



L. Friedman #243 (6-24-16)
mathplane.com

Blaise saw a pattern!



I. Expand the following:

SOLUTIONS

1) $(x + y)^5 =$ step 1: x terms $x^5 + x^4 + x^3 + x^2 + x^1 + x^0$
 step 2: y terms $y^0 + y^1 + y^2 + y^3 + y^4 + y^5$
 step 3: coefficients $\binom{5}{0} \binom{5}{1} \binom{5}{2} \binom{5}{3} \binom{5}{4} \binom{5}{5}$

2) $(m - p)^7 =$ $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

$m^7 - 7m^6p + 21m^5p^2 - 35m^4p^3 + 35m^3p^4 - 21m^2p^5 + 7m^6p - p^7$

3) $(b + 2)^4 =$ step 1: first terms --- $b^4 \quad b^3 \quad b^2 \quad b^1 \quad b^0$
 step 2: second terms --- $2^0 \quad 2^1 \quad 2^2 \quad 2^3 \quad 2^4$
 step 3: coefficients --- $\binom{4}{0} \binom{4}{1} \binom{4}{2} \binom{4}{3} \binom{4}{4}$
 combine and simplify --- $b^4 + 8b^3 + 24b^2 + 32b + 16$

4) $(2x - 3)^5 =$ first term: $\binom{5}{0}(2x)^5(-3)^0 = \frac{5!}{5!0!}(32x^5)(1)$
 second term: $\binom{5}{1}(2x)^4(-3)^1 = \frac{5!}{4!1!}(16x^5)(-3) = -15(16x^5)$
 etc.... $32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243$

5) $(x^2 + y^3)^6 =$ $x^{12} + 6x^{10}y^3 + 15x^8y^6 + 20x^6y^9 + 15x^4y^{12} + 6x^2y^{15} + y^{18}$

Binomial (Expansion) Theorem
 $(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{(n-k)} b^k$

II. Find the term:

using the formula:

1) $(x + 4y)^7$ 5th term $\binom{7}{5-1} x^{7-(5-1)}(4y)^{5-1}$
 $\binom{7}{4} x^3(4y)^4 = 35x^3(256y^4) = 8960x^3y^4$

2) $(3x - 2)^{12}$ 6th term find the pattern:
 1st term: $\binom{12}{0}(3x)^{12}(-2)^0$ the 6th term will be $\binom{12}{5} (3x)^7 (-2)^5$
 2nd term: $\binom{12}{1}(3x)^{11}(-2)^1$ $792 \quad 2187x^7 \quad -32$ $\rightarrow -55427328x^7$

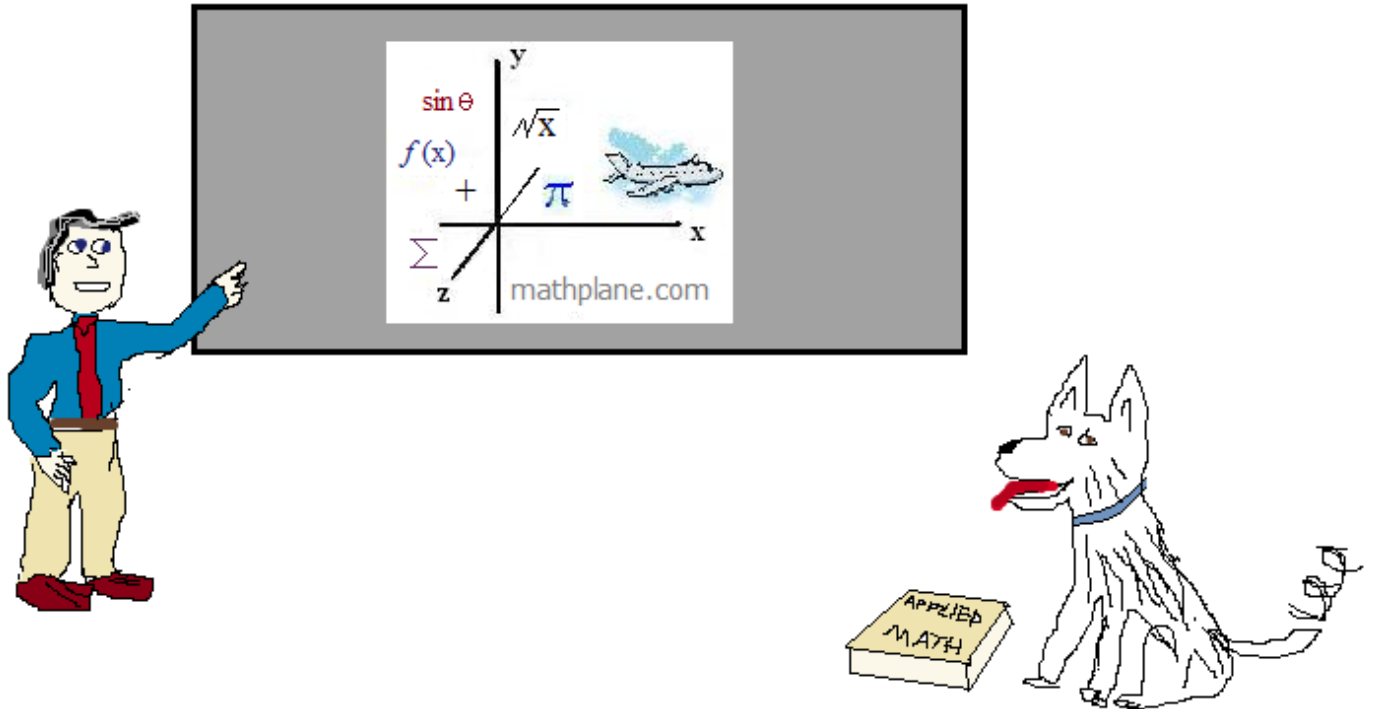
3) $(2m + p^2)^{24}$ 9th term
 $\binom{24}{8}(2m)^{16}(p^2)^8 = 735,471 \quad 65,536m^{16} p^{16}$
 $= 48,199,827,456m^{16}p^{16}$

The r^{th} term of the expansion $(a + b)^n$
 $\binom{n}{r-1} a^{n-(r-1)} b^{(r-1)}$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers



Mathplane *Express* for mobile and tablets at Mathplane.org

Also, at Facebook, Google+, Pinterest, TES, and TeachersPayTeachers