Double Integrals

(Notes, examples, and worksheet w/solutions)

Topics include multiple integrals, volume, integration by parts, and more.

Example: Calculate the volume of $2x + y - z = 0$ over the rectangle $R = (x, y)$ $\begin{matrix} 3 \le x \le 5 \\ 1 \le y \le 2 \end{matrix}$

We'll rewrite the volume equation: $z = 2x + y$

Then, construct the integrals....

$$\int_1^2 \int_3^5 2x + y \;\; dx\,dy$$

$$\left. x^2 + xy \right|_3^5 \;\Rightarrow\; 25 + 5y - 9 - 3y = 16 + 2y$$

$$\int_1^2 16 + 2y \;\; dy \;\Rightarrow\; \left. 16y + y^2 \right|_1^2 = 32 + 4 - 16 - 1 = \boxed{19}$$

Example: Find volume of tetrahedron under the plane $2x + y + z = 4$ and above the coordinate planes (in first octant)

$z = 4 - 2x - y$ the up and down partitions...

Then,

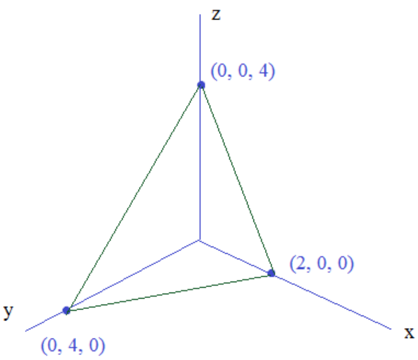
$y = -2x + 4$ the area partitions in the xy-plane

and,

$0 < x < 2$ the boundary of the area partitions

$$\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\,dy\,dx$$

$\boxed{16/3}$



$$\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\,dy\,dx$$

$$\int_0^2 \int_0^{4-2x} 4 - 2x - y \;\; dy\,dx$$

$$\int_0^2 (4 - 2x)(4 - 2x) - \frac{(4 - 2x)^2}{2} \;\; dx$$

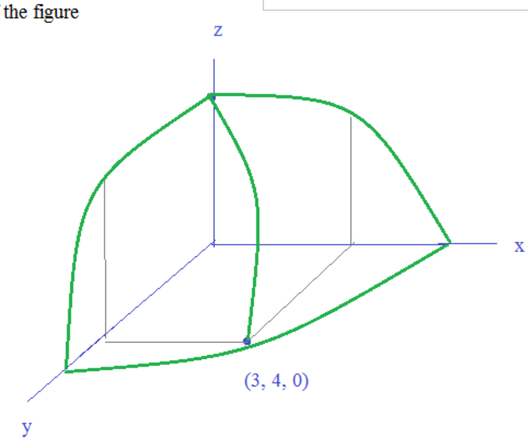
$$\int_0^2 \frac{(4 - 2x)^2}{2} \;\; dx$$

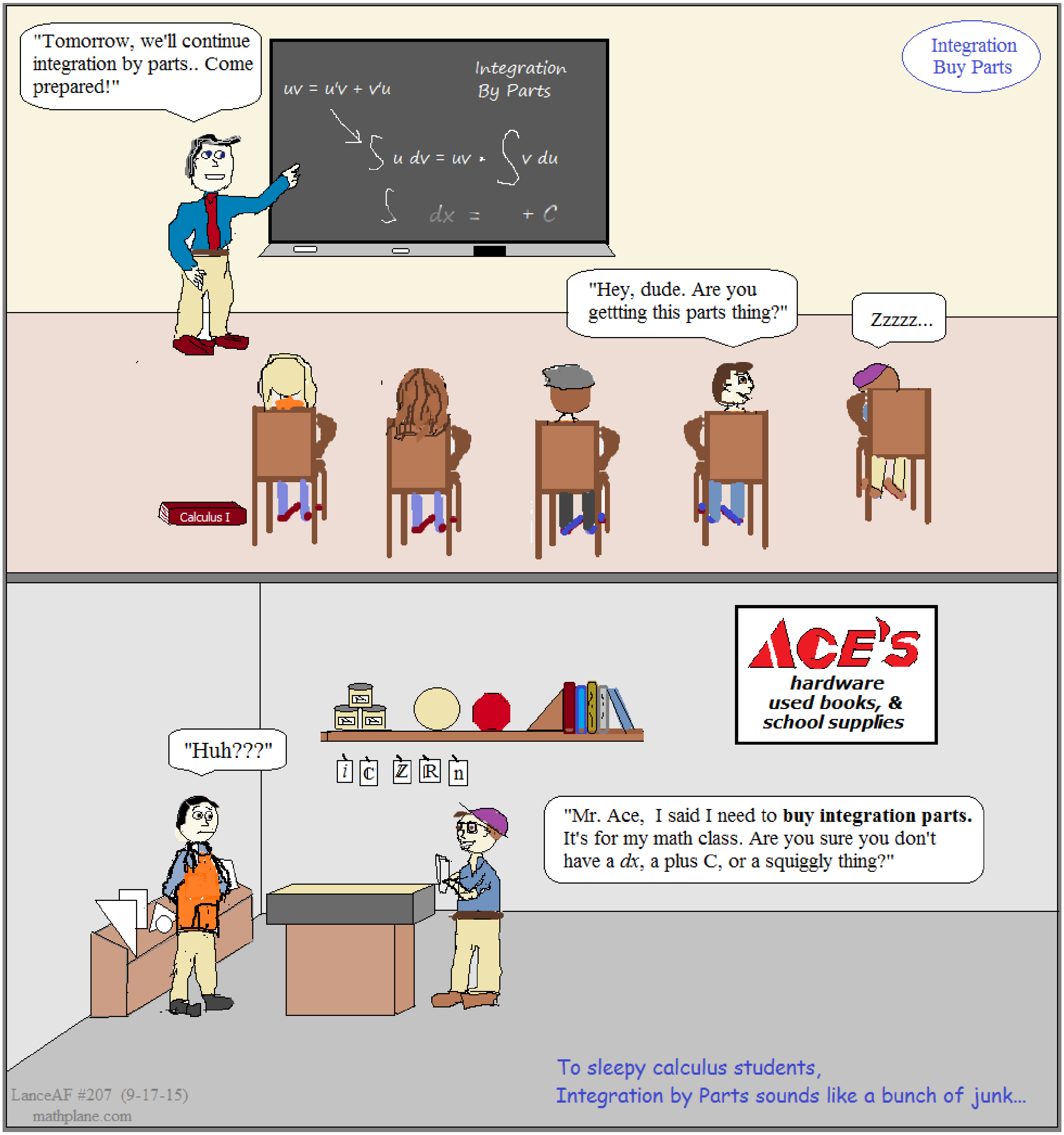
$$\left. \frac{-(4 - 2x)^3}{12} \right|_0^2 = \boxed{0 - (-64/12)}$$

Example: $$\int_0^4 \int_0^3 \sqrt{25 - x^2 - y^2} \;\; dx\;dy$$

Describe the shape and volume of the figure expressed in the double integral.

It describes the volume of a solid inscribed in a sphere with radius 5 in the 1st octant.




"Tomorrow, we'll continue integration by parts.. Come prepared!"
Integration By Parts
uv = u'v + v'u
∫ u dv = uv - ∫ v du
∫ dx = + C
Integration Buy Parts
"Hey, dude. Are you gettting this parts thing?"
Zzzzz...
Calculus I
ACE'S
hardware used books, & school supplies
"Huh???"
"Mr. Ace, I said I need to **buy integration parts.** It's for my math class. Are you sure you don't have a *dx*, a plus C, or a squiggly thing?"
To sleepy calculus students, Integration by Parts sounds like a bunch of junk...
LanceAF #207 (9-17-15)
mathplane.com

Worksheets-→

Double Integrals Worksheet

1) $$\int_{2}^{4}\int_{-1}^{1} x^2 + y^2 \; dy\, dx$$

2) $$\int_{0}^{1}\int_{0}^{1} \frac{1+x^2}{1+y^2} \; dy\, dx$$

3) $$\int_{0}^{1}\int_{0}^{s^2} \cos(s^3) \; dt\, ds$$

Double Integrals Worksheet

4) $$\int_0^1 \int_1^2 \frac{xe^x}{y}\, dy\, dx$$

5) $$\iint x\sin(y) - y\sin(x) \qquad 0 < x < \frac{\pi}{2}, \quad 0 < y < \frac{\pi}{3}$$

6) $$\iint_R x\cos(xy)\, dA$$ over the region $1 < x < 2$, $\frac{\pi}{2} < y < \pi$


"Does this look right?"
"The arms lengths aren't congruent."
"Huh?"
"My grandson loves math!"
I ♥ Grandm
LanceAF #367 (12-22-19)
mathplane.com
Ugly Sweaters
Hap-e Holidays
tanX
x
x^2
f(X)
Merry X-mas
Thoughtful -- and, colorful -- gifts for the holiday season?!?!

SOLUTIONS→

Double Integrals Worksheet

SOLUTIONS

1) $$\int_2^4 \int_{-1}^1 x^2 + y^2 \; dy\, dx$$

Integrate the inside first...

$$\int_{-1}^1 x^2 + y^2 \; dy \Rightarrow yx^2 + \frac{y^3}{3} \Bigg|_{-1}^{1} = (1)x^2 + \frac{(1)^3}{3} - \left((-1)x^2 + \frac{(-1)^3}{3} \right) = 2x^2 + \frac{2}{3}$$

Then, integrate the outside...

$$\int_2^4 2x^2 + \frac{2}{3} \; dx \Rightarrow \frac{2x^3}{3} + \frac{2}{3}x \Bigg|_2^4 = \frac{128}{3} + \frac{8}{3} - \left(\frac{16}{3} + \frac{4}{3} \right) = \boxed{\frac{116}{3}}$$

2) $$\int_0^1 \int_0^1 \frac{1 + x^2}{1 + y^2} \; dy\, dx$$

Integrate the inside first...
where x is the constant...

$$\int_0^1 \frac{1 + x^2}{1 + y^2} \, dy \Rightarrow (1 + x^2) \tan^{-1} y \Bigg|_0^1 = (1 + x^2) \frac{\pi}{4} - (1 + x^2)\, 0 = \frac{\pi}{4}(1 + x^2)$$

Then, integrate the outside...
where x is the variable...

$$\int_0^1 \frac{\pi}{4}(1 + x^2) \; dx \Rightarrow \frac{\pi}{4}\left(x + \frac{x^3}{3}\right) \Bigg|_0^1 = \frac{\pi}{4}\left(\frac{4}{3}\right) = \boxed{\frac{\pi}{3}}$$

Double Integrals Worksheet

SOLUTIONS

3) $$\int_0^1 \int_0^{s^2} \cos(s^3)\ dt\ ds$$

$$\int_0^1 t \cdot \cos(s^3) \Big|_0^{s^2} ds$$

$$\int_0^1 s^2 \cdot \cos(s^3) - 0 \cdot \cos(s^3)\ ds \implies \int_0^1 s^2 \cos(s^3)\ ds$$

$$\frac{1}{3}\sin(s^3) \Big|_0^1 = \boxed{\frac{1}{3}\sin(1)}$$

4) $$\int_0^1 \int_1^2 \frac{xe^x}{y}\ dy\ dx$$

Take first inner integral (where x is constant)

$$\int_1^2 \frac{xe^x}{y}\ dy \implies xe^x\ ln y \Big|_1^2 = xe^x\ ln(2) - xe^x\ ln(1) = ln(2)xe^x$$

Must apply integration by parts... $$\int_0^1 ln(2)\underbrace{x}_{u}\underbrace{e^x\ dx}_{dv}$$

$u \quad x \qquad v \quad e^x \qquad\qquad u\ dv = uv - \int v\ du$

$du \quad 1 \qquad dv \quad e^x$

$$ln(2)\left(xe^x - \int_0^1 e^x \right)$$

$$ln(2)\left(xe^x - e^x \Big|_0^1 \right) = \boxed{ln2}$$

Double Integrals Worksheet

SOLUTIONS

5) $\iint x\sin(y) - y\sin(x)$ $0 < x < \frac{\pi}{2}$, $0 < y < \frac{\pi}{3}$

We'll set up the definite integral, considering which part we want to integrate first.

$$\int_0^{\frac{\pi}{3}}\int_0^{\frac{\pi}{2}} x\sin(y) - y\sin(x)\ dx\,dy$$

(inside dx) $$\frac{x^2}{2}\sin(y) + y\cos(x)\Big|_0^{\frac{\pi}{2}} \Rightarrow \frac{\pi^2}{8}\sin(y) + y(0) - \left(\frac{0^2}{2}\sin(y) + y(1)\right) = \frac{\pi^2}{8}\sin(y) - y$$

(outside dy) $$\int_0^{\frac{\pi}{3}} \frac{\pi^2}{8}\sin(y) - y\ dy \Rightarrow \frac{\pi^2}{8}(-\cos(y)) - \frac{y^2}{2}\Big|_0^{\frac{\pi}{3}} = \frac{\pi^2}{8}\left(-\frac{1}{2}\right) - \frac{\pi^2}{18} - \left(\frac{\pi^2}{8}(-1) - 0\right)$$

$$\boxed{-\frac{\pi^2}{18} + \frac{\pi^2}{16}}$$

6) $\iint_R x\cos(xy)\ dA$ over the region $1 < x < 2$, $\frac{\pi}{2} < y < \pi$

$$\int_1^2\int_{\frac{\pi}{2}}^{\pi} x\cos(xy)\ dy\,dx$$

At first glance, it appears that integrating with respect to y first is easier...

$$\sin(xy)\Big|_{\frac{\pi}{2}}^{\pi} \Rightarrow \sin(\pi x) - \sin\left(\frac{\pi}{2}x\right)$$

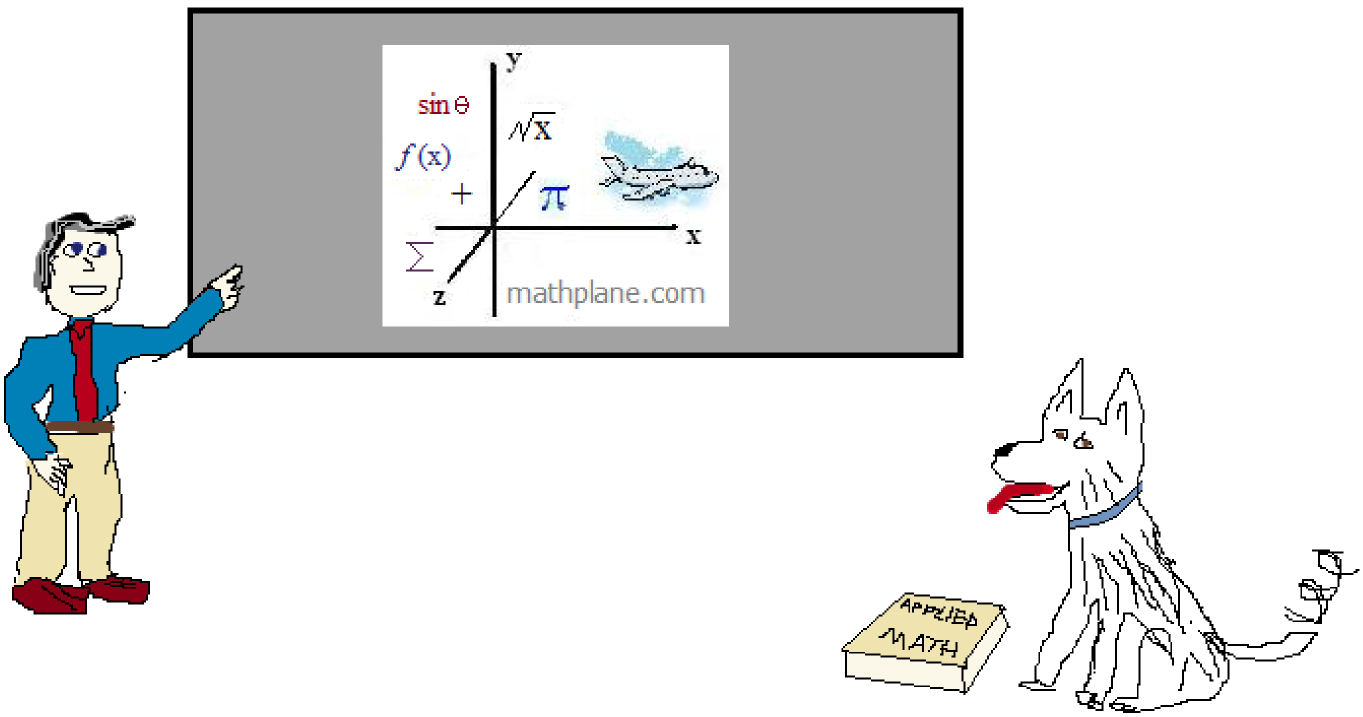
$$\int_1^2 \sin(\pi x) - \sin\left(\frac{\pi}{2}x\right)\ dx$$

$$-\frac{1}{\pi}\cos(\pi x) + \frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)\Big|_1^2 = -\frac{1}{\pi}(1) + \frac{2}{\pi}(-1) - \left(\frac{1}{\pi} + 0\right) = \boxed{-\frac{4}{\pi}}$$

Thanks for Visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers.



Also, at mathplane.ORG for mobile and tablets.

Or, visit our store at TeachersPayTeachers.