Word Problems: Interest, Growth/Decay, and Half-Life

Applying logarithms and exponential functions

Topics include simple and compound interest, *e*, depreciation, rule of 72, exponential vs. linear models, and more.

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Simple vs. Compound Interest

Observe the difference:

Simple Interest...

You deposit 10,000 into a bank that pays simple interest at a 10% interest rate

	t (years)	Home	Bank	Total
(bank deposit)	-----	10,000	0	10,000
	0	0	10,000	10,000
$1000 interest payments every year are sent home	1	1000	10,000	11,000
	2	2000	10,000	12,000
	3	3000	10,000	13,000
	4	4000	10,000	14,000

Compound Interest...

You deposit $10,000 into a bank that earns 10% interest *compounded* annually.

Total	Bank	Home	t (years)	
10,000	0	10,000	-----	(bank deposit)
10,000	10,000	0	0	
11,000	11,000	0	1	The 10% interest payments every year stay in the bank!
12,100	12,100	0	2	
13,310	13,310	0	3	
14,641	14,641	0	4	

Interest Formulas

Simple Interest

$$A = P + P\,r\,t$$

$$A = P + I$$

$$I = P\,r\,t$$

A = future amount
P = principal amount
r = annual interest rate
t = time (years)
I = simple interest earned

Compound Interest

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

A = future amount
P = principal amount
r = (nominal) annual interest rate
n = number of times of compounding
t = years

note: nt = number of times the amount is compounded

$\frac{r}{n}$ = the rate that is paid at each compounding time.

Compound Interest comparisons

Assume 12% *annual* interest rate: $1000 deposit....

compounded	annually	semi-annually	Quarterly	Monthly	Daily	Continuously
# of payments	1	2	4	12	365	----
rate of each payment	12%	6%	3%	1%	.0328%	$A = Pe^{rt}$
Future amount	1120	1123.60	1124.86	1126.83	1127.47	1127.50
1 year earnings	120	123.60	124.86	126.83	127.47	127.50

**The more times the principal (and interest) compounds, the more the amount grows!!

"The Rule of 72"

What is it? If a sum is compounding at X%, that sum will double in approximately $\frac{72}{X}$ cycles.

Example:

Suppose you deposit \$100 in a bank that earns 10% interest per year. How many years will it take to double your money?

Using the "rule of 72"

$$\frac{72}{10} = 7.2$$

So, a little over 7 years...

time	balance
0	100 (initial deposit)
1	110 (100 + 10)
2	121 (110 + 11)
3	133.10 (121 + 12.10)
4	146.41 (133.1 + 13.31)
5	161.05 (146.41 + 10% of 146.41)
6	177.15 (161.05 + interest payment)
7	194.86 (177.15 + interest on 177.15)
8	212.57

Account doubles from 100 to 200 shortly after 7 years...

Formula for compounding interest:

$$\text{Amount} = P\,(1 + \text{interest rate})^t$$

(where P is the principle and t is the time it compounds)

Using our example:

$$200 = 100\,(1 + .10)^t$$

$$\frac{200}{100} = (1.10)^t$$

$$2 = (1.10)^t$$

(Use a calculator to solve. Or, use logarithms)

$$\log 2 = \log (1.10)^t$$

$$\log 2 = t \bullet \log (1.10)$$

$$t = \frac{\log 2}{\log (1.10)} = \frac{.3010}{.0414} \cong 7.27$$

**If you're earning 10% interest, it will take approximately 7.27 years for you to double your money!

Exponential Growth Applications

Example: A cell population t days from now is modeled by $A(t) = 8e^{0.5t}$

1) What is the current cell population?

The current population is when time (t) = 0 $A(0) = 8e^{0.5(0)} = \boxed{8}$ (also, the *initial amount*)

2) Determine the cell population 5 days from now.

And, the cell population 5 days from now, occurs when t = 5 $A(5) = 8e^{0.5(5)} = 8 \cdot 12.18 = \boxed{97.46}$

3) When will the cell population exceed 200?

Let the amount = 200.. then solve (using logarithms)

$$200 = 8e^{0.5t}$$

$$25 = e^{0.5t}$$

$$\ln 25 = \ln e^{0.5t}$$

$$\ln 25 = 0.5t \cdot \ln e$$

$$3.219 = 0.5t$$

$$\boxed{t = 6.44 \text{ days}}$$

Example. Stan invests \$10,000 in an account that earns 7% compounded annually.
Eric invests \$8,000 in an account that earns 9% compounded annually.

How many years will it take before Eric has more money in his account?

$$A = P(1 + r)^t$$

Stan's account: $A = 10,000(1 + .07)^t$

Eric's account: $A = 8,000(1 + .09)^t$

Since Eric's account is increasing at a faster rate,
eventually it'll exceed Stan's...
So, let's find when they are equal....

$$10,000(1 + .07)^t = 8,000(1 + .09)^t$$

$$\frac{5}{4}(1.07)^t = (1.09)^t$$

$$\frac{5}{4} = \frac{(1.09)^t}{(1.07)^t}$$

$$\log \frac{5}{4} = \log \left(\frac{1.09}{1.07}\right)^t$$

$$\log \frac{5}{4} = t \cdot \log \left(\frac{1.09}{1.07}\right)$$

$$t = \frac{\log \frac{5}{4}}{\log \left(\frac{1.09}{1.07}\right)} = \frac{.0969}{.0080} = \boxed{12.05 \text{ years}}$$

Check:

12 years (Stan)

$10,000(1.07)^{12.1} = \$22,674.81$

12 years (Eric)

$8,000(1.09)^{12.1} = \$22,696.07$

Example: Ian Apple has $750 in his savings account that earns 2.5% compounded continuously. The cost of a phone he likes costs $500. But, the price of the phone is increasing continously at a rate of 7%.

When will Mr. Apple no longer have enough savings to buy the phone?

Savings = $750\,e^{.025t}$

phone cost = $500\,e^{.07t}$

set them equal to determine year they match...

$750\,e^{.025t} = 500\,e^{.07t}$

$\frac{3}{2}\,e^{.025t} = e^{.07t}$

$1.5 = e^{.045t}$

'raise each to the ln'...

$\ln(1.5) = \ln e^{.045t}$

$\ln(1.5) = .045t(\ln e)$

$\ln(1.5) = .045t$

$\boxed{\text{9.01 years...}}$

Example: $20,000 is deposited into an investment account that offers a 6% interest rate.

How much will the account have after 6 1/2 years compounded

a) semi-annually? b) monthly? c) continuously?

Amount = Principal$(1 + \frac{\text{rate}}{\text{n}})^{\text{nt}}$

n = # of payments per year..
t = # of years

Amount = Pe^{rt}

P = Principal r = interest rate t = time

a) Amount = $20{,}000(1 + \frac{.06}{2})^{2(6.5)}$

$= 20{,}000(1.03)^{13}$

$= \boxed{\$29{,}370.67}$

b) Amount = $20{,}000(1 + \frac{.06}{12})^{12(6.5)}$

$= 20{,}000(1.005)^{78}$

$= \boxed{\$29{,}510.92}$

c) Amount = $20{,}000e^{.06(6.5)}$

$= 20{,}000e^{.39}$

$= \boxed{\$29{,}539.62}$

Example: Ancient paintings are discovered in a remote cave.
To estimate the age, scientists take a sampling and measure the carbon count, knowing that the decay of carbon is

$$A = A_0\, e^{-.0001201\,t}$$

If the sample contains 15% of the original carbon-14, estimate how old the painting is.

Note: the half-life of carbon is 5770 years.

$1/2 = e^{r(5770)}$

$\ln(.5) = 5770r$

$r = -.0001201$

we'll suppose the original sample was 100...
If 15% remains, then

$15 = 100\,e^{-.0001201\,t}$

$.15 = e^{-.0001201\,t}$

$\ln(.15) = \ln e^{-.0001201\,t}$

$\ln(.15) = -.0001201t \cdot (\ln e)$

$\boxed{t \cong 15{,}796 \text{ years...}}$

(Note: without rounding, the time will be closer to approx. 15, 678 years.)

Example: RC Adams has $2500 in a savings account that earns 3.5% compounded continuously. The current price of a screen tv he likes costs $6000. But, the cost of the tv is *decreasing* continuously at a rate of 7%

When will RC A. be able to afford the television?

RCA's savings will increase... Meanwhile, the television will decrease...
At some point, they will be equal (and RCA will be able to buy the tv!)

RCA money = $\$2500e^{.035t}$

TV cost = $\$6000e^{-.07t}$

set them equal....

$\$2500e^{.035t} = \$6000e^{-.07t}$

$e^{.035t} = 2.4e^{-.07}$

$e^{.105t} = 2.4$

$\ln(2.4) = .105t$

$t = \frac{\ln(2.4)}{.105} = \boxed{\text{8.34 years...}}$

Linear vs. Exponential Model

Example: Given the coordinates (0, 100) and (2, 400)

Find a *linear* equation that passes through the coordinates.

Find an *exponential* equation that goes through the coordinates.

Graph both functions.

To find a linear equation, you need the slope and a point.

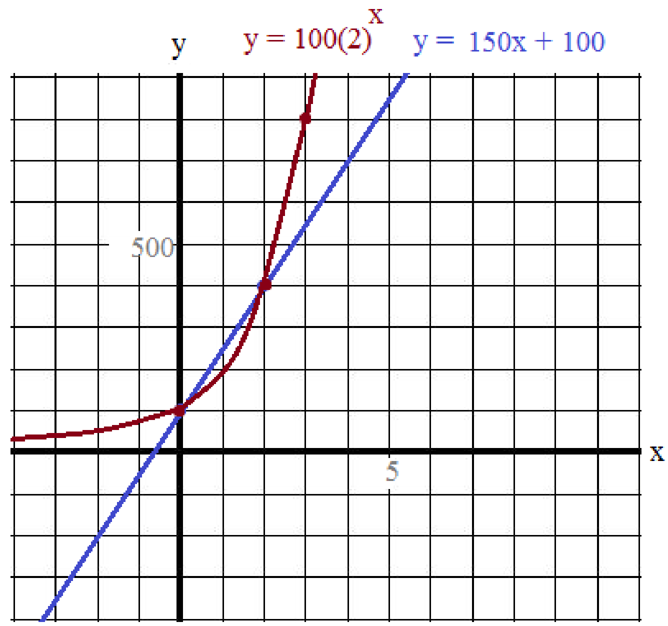
Slope:

$$\frac{\text{"rise"}}{\text{"run"}} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{400 - 100}{2 - 0} = 150$$

A point: (0, 100)

Equation of the line: $y - 100 = 150(x - 0)$

or $y = 150x + 100$



To find the exponential equation, use the general form $y = ab^x$

Substitute (0, 100): $100 = ab^0$

$100 = a(1)$

$a = 100$

Substitute (2, 400): $400 = ab^2$

$400 = (100)b^2$

$b^2 = 4$

$b = 2$

Equation of the exponential curve: $y = 100(2)^x$

Example:

In January 1923, 79,000 residents lived in Emigration City, USA.
Each year, 1/3 of the population would leave. (Assuming there are no births, deaths, or immigrants,) what year did the population eventually fall to under 100 residents?

If 1/3 leave each year, then 2/3 remain...

We can model the population of Emigration City.

$$\boxed{\begin{array}{l} 79{,}000(2/3)^n = \text{Population} \\ \text{where } n = \text{number of years after 1923} \end{array}}$$

$$79{,}000(2/3)^n = 100$$

$$(2/3)^n = \frac{100}{79{,}000}$$

$$n\log(2/3) = \log \frac{100}{79{,}000}$$

$$n = \frac{\log \frac{100}{79{,}000}}{\log(2/3)} = \frac{-2.898}{-0.176} = 16.46 \text{ (approximately)}$$

Therefore, in the 17th year, the population will fall under 100.

$$\boxed{1939}$$

Or, calculating 1 year at a time:

(population at the beginning of each year)

1st year ⟶	1923:	79,000	
	1924:	52667	52666
	1925:	35111	35110
	1926:	23408	23406
	1927:	15605	15604
	1928:	10404	10402
	1929:	6936	6934
	1930:	4624	4622
	1931:	3083	3081
	1932:	2056	2054
	1933:	1371	1369
	1934:	914	912
	1935:	610	608
	1936:	407	405
	1937:	272	270
	1938:	182	180
17th year ⟶	1939:	122	120
	1940:	82	80

(first column is rounding up.. second column is rounding down...)

Practice Questions (and Answers) -→

Growth, Decay, and Interest Rates

1) A bank offers 4.25% interest rate *compounded daily*. What is the annual yield?

2) A bacteria grows exponentially.
The initial amount was 10,000. After 1 hour, the amount grew to 25,000.

a) Find the doubling period. (How long does it take the bacteria to double?)

b) What is the amount after 3 hours?

3) A special element has a half-life of 5 days. The initial amount is 500 grams.

a) How many grams will remain after 15 days?

b) What is the daily rate of decay?

c) How many grams will remain after 12 days?

Growth, Decay, and Interest Rate Questions

4) A radioactive element has a half-life of 1000 years.
What percentage of the sample remains after 250 years?

5) If 250 mg of a radioactive material decays to 200 mg in 48 hours,
what is the 1/2 life of the material?

6) Jim deposits $1000 into a bank that pays 8% annual interest, compounded continuously.
Carol deposits $250 at the beginning of each quarter (Jan., Apr., July, Oct.) into the same bank.

At the end of the year, how much more will Jim have in his account?

7) Compounding and Investing...

A) Scenario #1: invest $5,000 at age 25... 8% interest compounded annually

Scenario #2: invest $20,000 at age 45... 8% interest compounded annually

Which will have more money at age 65?

B) Scenario #3: invest $1000 per year from age 21 to age 30 (10 years)

Scenario #4: invest $1000 per year from age 31 to age 65 (35 years)

assume 8% interest compounded annually

Which will have more money at age 65? Explain..

8) Depreciating.... A car cost $43,500. The car's value depreciates at a rate of 4%.

A) How much is the car worth after 7 years if the rate decreases

a) annually?

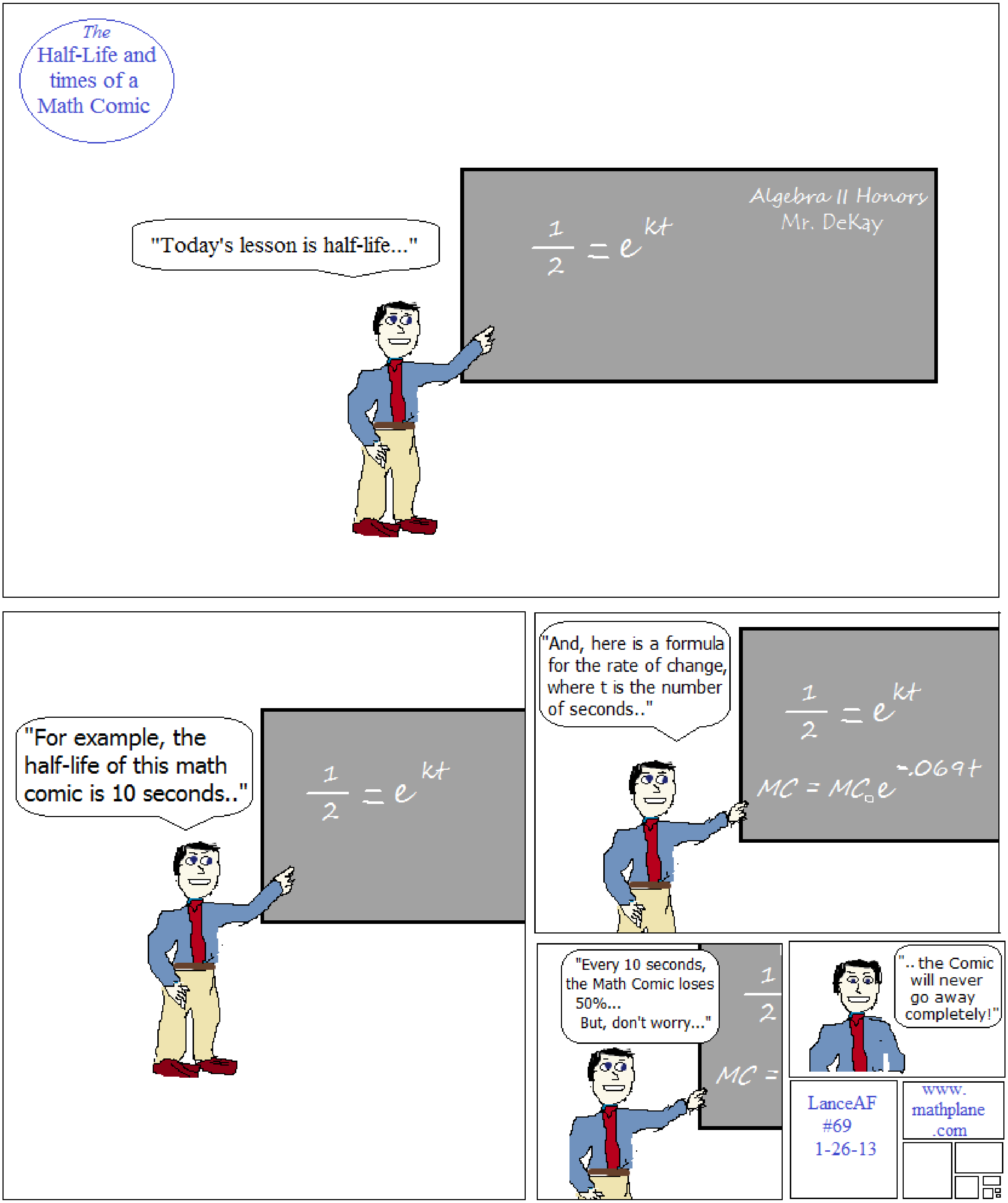
b) quarterly?

c) continuously?

B) How much was the car worth 5 years ago, if the depreciation occurred

a) quarterly?

b) continuously?


The Half-Life and times of a Math Comic
"Today's lesson is half-life..."
Algebra II Honors Mr. DeKay
$\frac{1}{2} = e^{kt}$
"For example, the half-life of this math comic is 10 seconds.."
$\frac{1}{2} = e^{kt}$
"And, here is a formula for the rate of change, where t is the number of seconds.."
$\frac{1}{2} = e^{kt}$
$MC = MC_0 e^{-.069t}$
"Every 10 seconds, the Math Comic loses 50%... But, don't worry..."
$\frac{1}{2}$
$MC =$
".. the Comic will never go away completely!"
LanceAF #69 1-26-13
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Growth, Decay, and Interest Rates

1) A bank offers 4.25% interest rate *compounded daily*. What is the annual yield?

Suppose we deposit \$10000. What is the amount one year later?

$A = P(1 + \frac{r}{n})^{nt}$

A = future amount
P = principal amount
r = interest rate
n = number of times the amount is compounded per year
t = number of years

$A = 10000(1 + \frac{.0425}{365})^{365(1)} = 10000(1.00011644)^{365} = 10,434.13$

the rate of *each* compounding

the "number of compoundings"

So, \$10,000 to \$10,434 is an increase of \$434 in one year....

$\frac{434}{10,000} = .0434$

4.34% annual yield...

2) A bacteria grows exponentially.
The initial amount was 10,000. After 1 hour, the amount grew to 25,000.

a) Find the doubling period. (How long does it take the bacteria to double?)

b) What is the amount after 3 hours?

Compounding exponentially:

$y = ab^x$

We are given 2 points of measurement: (0, 10,000) and (1, 25,000)

Use substitution: $y = ab^x$

$(10,000) = ab^{(0)}$

$10,000 = a(1)$

$a = 10,000$

then, $(25,000) = ab^{(1)}$

$25,000 = 10,000b$

$b = 2.5$

The bacteria equation is $y = 10000(2.5)^x$

a) At the end of the doubling period, the bacteria amount will be 20,000...

$20,000 = 10,000(2.5)^x$

$2 = (2.5)^x$

$\log 2 = \log(2.5)^x$

$.301 = x\log(2.5)$

$.301 = .398x$

$\boxed{x = .756 \text{ hours or } 45.4 \text{ minutes}}$

b) After 3 hours,

$y = 10000(2.5)^3 = \boxed{156,250}$

3) A special element has a half-life of 5 days. The initial amount is 500 grams.

a) How many grams will remain after 15 days?

b) What is the daily rate of decay?

c) How many grams will remain after 12 days?

a)

Days	amount
0	500
5	250
10	125
15	62.5
20	31.25

62.5 grams

Finding Half life formula:

$A = Pe^{rt}$

A = final amount
e = euler's number
r = rate
t = time

$\frac{1}{2}P = Pe^{rt}$

$\frac{1}{2} = e^{rt}$

b) $250 = 500e^{5r}$

$\frac{1}{2} = e^{5r}$

$\ln\frac{1}{2} = \ln e^{5r}$

$-.693 = 5r(\ln e)$

$-.693 = 5r(1)$

$r = -.1386$

$\boxed{\text{daily rate: } -13.86\%}$

c) $A = 500(e)^{(-.1386)(12)}$

$A = 500(.19)$

approximately 95.25 grams..

Growth, Decay, and Interest Rate Questions

SOLUTIONS

4) A radioactive element has a half-life of 1000 years.
What percentage of the sample remains after 250 years?

$\boxed{A = Pe^{rt}}$

Part 1: Find the rate

$$\frac{1}{2} = e^{rt}$$

$$\frac{1}{2} = e^{1000r}$$

$$\ln\frac{1}{2} = \ln e^{1000r}$$

$$-.693 = 1000r$$

$$r = -.000693$$

Part 2: Apply formula

$$A = Pe^{250(-.000693)}$$

$$A = (1)e^{250(-.000693)}$$

$$A = .841$$

$\boxed{\text{Approximately 84.1\% remains}}$

5) If 250 mg of a radioactive material decays to 200 mg in 48 hours,
what is the 1/2 life of the material?

first, find the rate of decay...

$$A = Pe^{rt}$$

$$200 = 250e^{r(48)}$$

$$.8 = e^{48r}$$

$$r = -.0046$$

then, find the 1/2 life

$$125 = 250e^{-.0046(t)}$$

$$.5 = e^{-.0046(t)}$$

$$\boxed{t = 149 \text{ hours}}$$

6) Jim deposits \$1000 into a bank that pays 8% annual interest, compounded continuously.
Carol deposits \$250 at the beginning of each quarter (Jan., Apr., July, Oct.) into the same bank.

At the end of the year, how much more will Jim have in his account?

$A = Pe^{rt}$

Jim: $A = 1000(e)^{.08(1)} = \boxed{1083.287}$

Total: \$1083.29

Carol: $A = 250(e)^{.08(1)} = 270.822$

first deposit

$A = 250(e)^{.08(.75)} = 265.459$

second deposit (compounds for 9 months)

$A = 250(e)^{.08(.50)} = 260.203$

third deposit (compounds for 6 months)

$A = 250(e)^{.08(.25)} = 255.050$

fourth deposit (compounds for 3 months)

Total: $\boxed{\$1051.53}$

Jim will have \$31.76 more at the end of the year...

7) Compounding and Investing...

A) Scenario #1: invest \$5,000 at age 25... 8% interest compounded annually

Scenario #2: invest \$20,000 at age 45... 8% interest compounded annually

SOLUTIONS

Which will have more money at age 65?

$\boxed{\text{Amount 1} = 5{,}000(1+.08)^{40} = \$108{,}622}$

$\text{Amount 2} = 20{,}000(1+.08)^{20} = \$93{,}219$

Note the powering of compounding interest! (Also, notice how investing early is important!)

then, recognize the "rule of 72" ---- at 8%, you would expect your money to roughly double every $\frac{72}{8} = 9$ years

Scenario #1 would double the money almost 4 1/2 times...

5K, 10K, 20K, 40K, 80K, 120K (doubling 5 times)

Scenario #2 would double the money twice and a bit more...

20K, 40K, 80K (doubling twice)

B) Scenario #3: invest \$1000 per year from age 21 to age 30 (10 years)

Scenario #4: invest \$1000 per year from age 31 to age 65 (35 years)

assume 8% interest compounded annually

Which will have more money at age 65? Explain..

Scenario #3 segments

$\text{Amount} = 1000(1.08)^{44} = 29{,}555$ (age 21 deposit)

$= 1000(1.08)^{40} = 21{,}724$ (age 25 deposit)

$= 1000(1.08)^{35} = 14{,}785$ (age 30 deposit)

Scenario #4 segments

$\text{Amount} = 1000(1.08)^{34} = 13{,}690$ (age 31 deposit)

$= 1000(1.08)^{17} = 3{,}700$ (age 48 deposit)

$= 1000(1.08)^{1} = 1{,}080$ (age 64 deposit)

Apparently, $\boxed{\text{Scenario \#3 will have more money...}}$

Using a rough estimate, Scenario #3 will end up with about 200K;

and, Scenario #4 will end up with about 175K..

8) Depreciating.... A car cost \$43,500. The car's value depreciates at a rate of 4%.

A) How much is the car worth after 7 years if the rate decreases

a) annually? $V = 43{,}500(1-.04)^{7} = \boxed{\$32{,}688}$

b) quarterly? $V = 43{,}500\left(1-\frac{.04}{4}\right)^{7(4)} = 43{,}500(1-.01)^{28} = \boxed{\$32{,}830}$

c) continuously? $V = 43{,}500e^{-.04(7)} = \boxed{\$32{,}876}$

Notice: when you increase the number of depreciations, the depreciation/decay slows!
EX: taking 1% each quarter will decrease LESS than taking 4% at the end of the year!

B) How much was the car worth 5 years ago, if the depreciation occurred

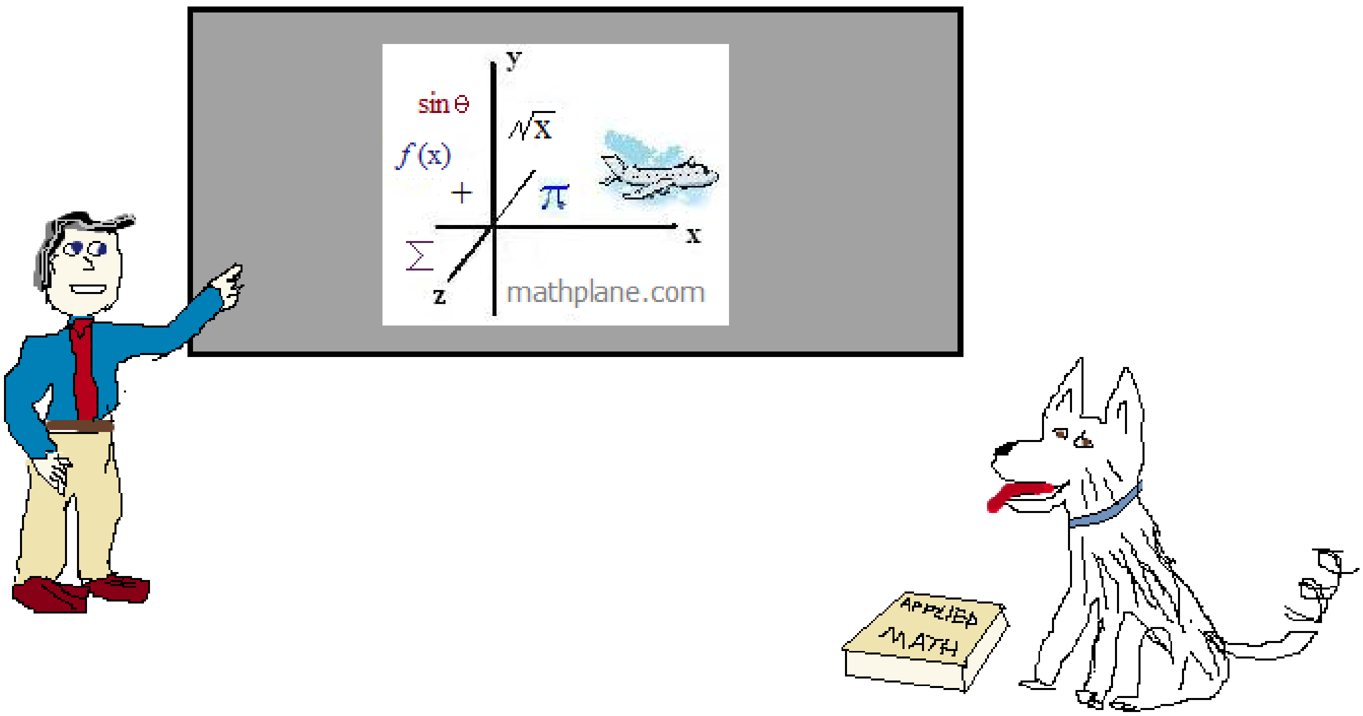
a) quarterly? $V = 43{,}500\left(1-\frac{.04}{4}\right)^{-5(4)} = 43{,}500(.99)^{-20} = \boxed{\$53{,}184}$

b) continuously? $V = 43{,}500e^{-.04(-5)} = \boxed{\$53{,}131}$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers



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