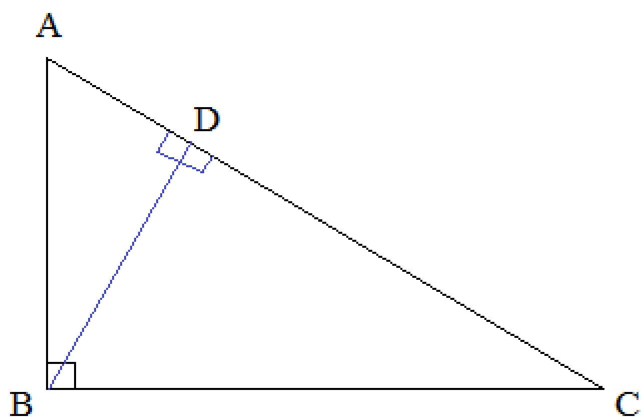
Similar Triangles and Ratios

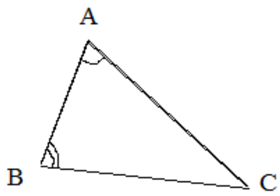
Notes, Examples, and Practice Test (w/solutions)



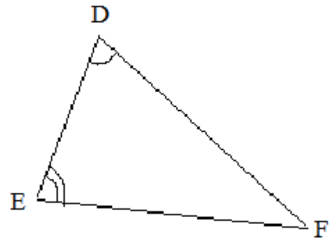
This introduction includes similarity theorems, geometric means, side-splitter theorem, angle bisector theorem, mid-segments, and more.

Similar triangles: Angle - Angle

Definition: If a pair of <u>corresponding angles</u> of 2 triangles are <u>congruent</u>, then the triangles are <u>similar</u>.



$\triangle ABC \sim \triangle DEF$



Comments:

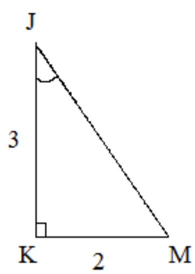
1) $\triangle ABC \sim \triangle DEF$ --- the angles should be expressed in proper order to indicate which angles are congruent.

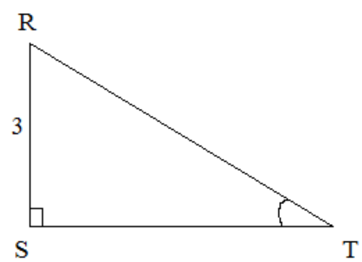
2) Angles C and F can easily be proven congruent by substitution:

Statements	Reasons
1) $\angle A = \angle D$, $\angle B = \angle E$	1) Given
2) $A + B + C = 180^\circ$, $D + E + F = 180^\circ$	2) Sum of interior angles of triangle is 180 degrees
3) $F = 180^\circ - (D + E)$, $C = 180^\circ - (A + B)$	3) Subtraction
4) $C = 180^\circ - (D + E)$	4) Subtstitution
5) $C = F$	5) Substitution

3) The ratios of the corresponding sides will be equal; and, the ratio of the perimeter will be consistent with the sides.

Example:





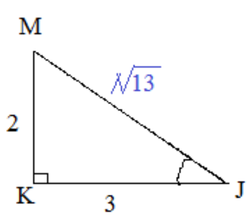
Since K and S are right angles, they are congruent...

and, since J and T are congruent,

$\triangle MKJ \sim \triangle RST$

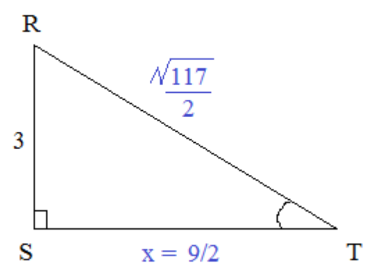
(note: the triangles are expressed in 'corresponding order')

(JKM is rotated and reflected to visually correspond to RST)



perimeter: 8.6 units

area: 3 sq. units



perimeter: 12.9 units

area: 6.75 sq. units

$$\frac{MK}{RS} = \frac{KJ}{ST}$$

$$\frac{MK}{RS} = \frac{2}{3} \qquad \frac{KJ}{ST} = \frac{3}{x}$$

$$\frac{2}{3} = \frac{3}{x} \qquad x = 9/2$$

ratio of small to large triangle is 2:3

ratio of perimeters is 2:3

ratio of the areas is 4:9

$(2^2 : 3^2)$

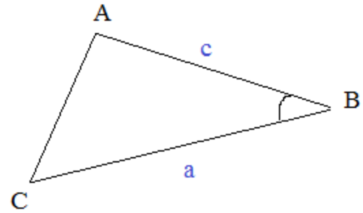
Similar triangles: Side - Angle - Side

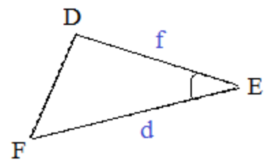
Definition: If a pair of <u>corresponding sides</u> of 2 triangles have the same ratio AND the <u>included</u> angles are congruent, then the triangles are <u>similar</u>.

$\triangle ABC \sim \triangle DEF$

$\angle B = \angle E$

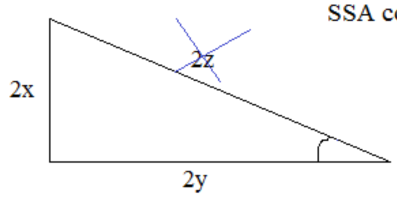
$\frac{c}{f} = \frac{a}{d}$

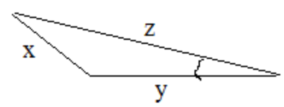




Comments: 1) SAS: it must be the *included* angle!

SSA counter-example:





ratio of 2 sides are equal, & non-included angles are congruent -- but, triangles are not similar!

2) The ratio between 2 sides of one triangle will be identical to the ratio of the corresponding 2 sides of the other triangle

$$\frac{c}{f} = \frac{a}{d} \longrightarrow af = cd \longrightarrow \frac{af}{c} = d \longrightarrow \frac{a}{c} = \frac{d}{f}$$

3) $\triangle ABC \sim \triangle DEF$ --- the angles should be expressed in proper order to indicate which angles are congruent.

Example: Using SAS, verify that $\triangle JKL \sim \triangle MKP$

small triangle	KM = 4 units	KP = 6 units	$\angle K = 90^\circ$
large triangle	KJ = 6 units	KL = 9 units	$\angle K = 90^\circ$
	2:3 ratio	2:3 ratio	congruent

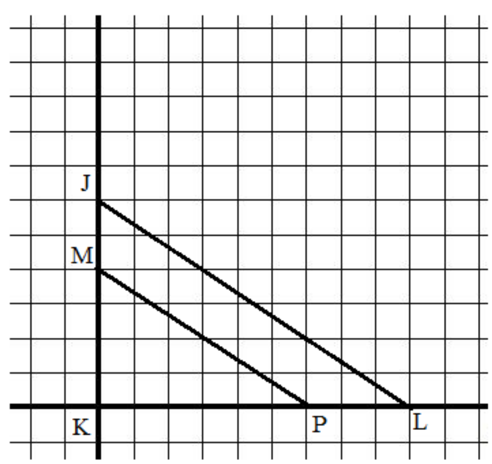
similar triangles.....

(using pythagorean theorem or distance formula)

$JL = \sqrt{117}$

$MP = 2\sqrt{13}$

$\frac{2\sqrt{13}}{\sqrt{117}} = \frac{2}{3}$ ✓



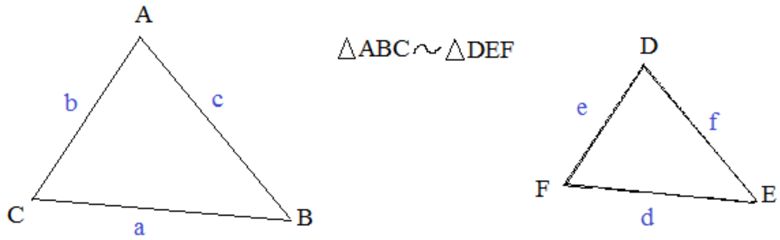
And, since the triangles are similar, the corresponding angles are congruent, and <u>therefore,</u>

$\overline{JL}$ is parallel to $\overline{MP}$

(Parallel lines cut by transversals)

Similar triangles: Side - Side - Side

Definitions: If the (three) corresponding sides of 2 triangles are proportional, then the triangles are similar.



$\triangle ABC \sim \triangle DEF$

$$\frac{b}{e} = \frac{c}{f} = \frac{a}{d}$$

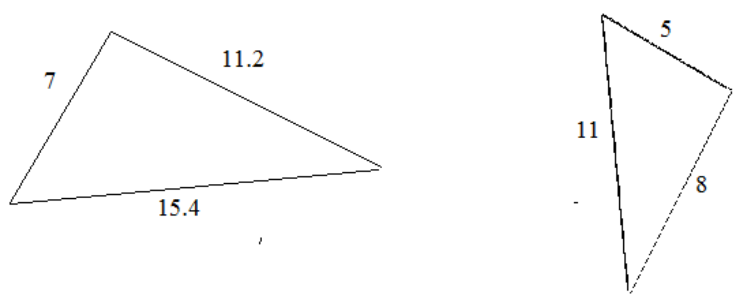
Comments: 1) If $xf = c$, then the perimeter of $\triangle ABC = x$(perimeter of $\triangle DEF$)

2) Using trigonometry, the angles can be determined from the sides. (and verified to be congruent)

3) SSS congruency vs. SSS similarity:

If 3 corresponding sides are *proportional*, the triangles are *similar*;
If 3 corresponding sides are *congruent*, then the triangles are *congruent* (the ratio is 1)

Example: Do the following triangles have the same shape? (are they similar?)



Since we are given the sides, let's compare each pair of sides:

a) smallest sides (opposite smallest angles)

$$\frac{7}{5} = 1.4$$

b) medium sides (opposite middle angles)

$$\frac{11.2}{8} = 1.4$$

c) largest sides (opposite largest angles of each triangle)

$$\frac{15.4}{11} = 1.4$$

The ratio of each pair of sides is 1:1.4

Therefore, the triangles are similar!

Similar Triangle Geometry Problems

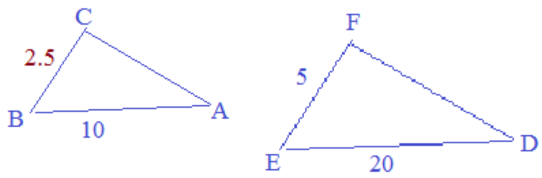
Given: $\triangle ABC \sim \triangle DEF$

$\overline{AB} = 10$
$\overline{DE} = 20$
$\overline{EF} = 5$

Find the length of $\overline{BC}$

Solution:

Step 1: Draw a picture



Step 2: Identify proportions/ratios

$$\frac{BC}{EF} = \frac{AB}{DE}$$

$$\frac{BC}{5} = \frac{10}{20}$$

Step 3: Solve (cross multiply)

$$20(BC) = 5(10)$$

$$\overline{BC} = 2.5$$

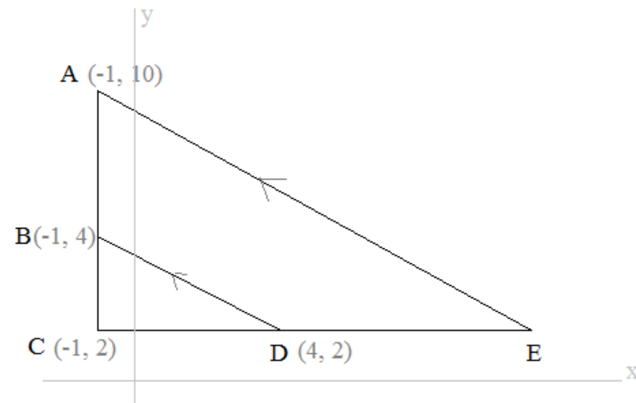
Given: $\overline{AE} \parallel \overline{BD}$

Answer the following:

1) What are the similar triangles? (explain why)

2) Find coordinate of E

3) Find and compare the lengths of $\overline{AE}$ and $\overline{BD}$.



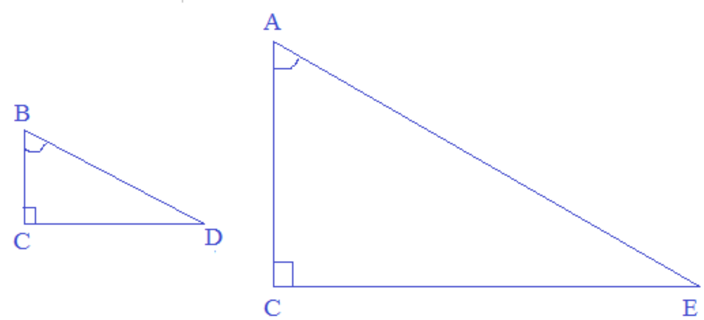
Solutions:

1) $\triangle ACE \sim \triangle BCD$

Angle-Angle theorem: If a pair of corresponding angles are congruent, then the triangles are similar.

$\angle A = \angle B$ (parallel lines cut by a transveral)

$\angle C = \angle C$ (reflexive property)



2) Since the triangles are similar, we can use proportions/ratios to find the other coordinate (and lengths).

$$\frac{BC}{AC} = \frac{CD}{CE}$$

$$\frac{2}{8} = \frac{5}{x}$$

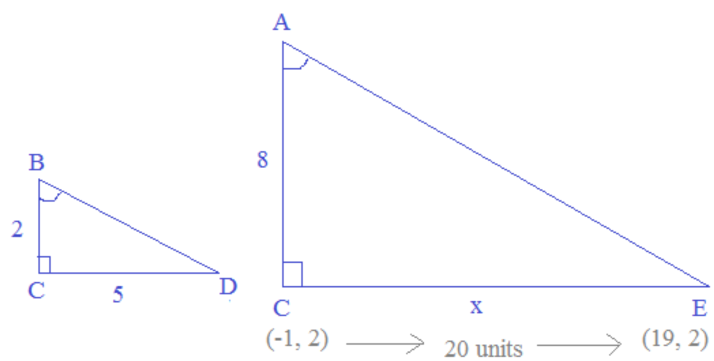
$$2x = 40$$

$$x = 20$$

Since C is at (-1, 2),

a horizontal move 20 units to the right to point E

→ (19, 2)



3) Using the pythagorean theorem,

$$BD^2 = BC^2 + CD^2$$

$$= 4 + 25$$

$$\overline{BD} = \sqrt{29}$$

**Since the ratio of the triangles is 1:4, the length of AE should be $4\sqrt{29}$

$$AC^2 + CE^2 = AE^2$$

$$64 + 400 =$$

$$\overline{AE} = \sqrt{464} = 4\sqrt{29}$$ ✓

Similar Triangle Word Problems

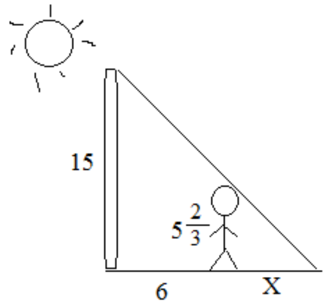
Solving "similar triangle/ratio" problems:

1) Draw picture
2) Split triangles
3) Solve proportion
4) Check answer

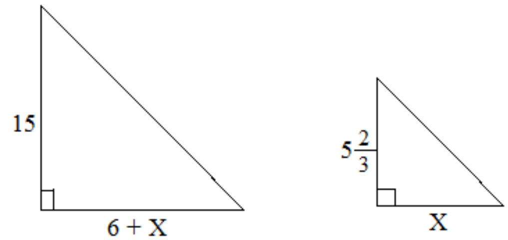
Example: A 5'8" person stands 6 feet from a 15-foot tall lamp post. If their shadows overlap, how long is the person's shadow?

Solution:

Step 1: Draw a Picture



Step 2: Split the triangles



Step 3: Solve the proportion

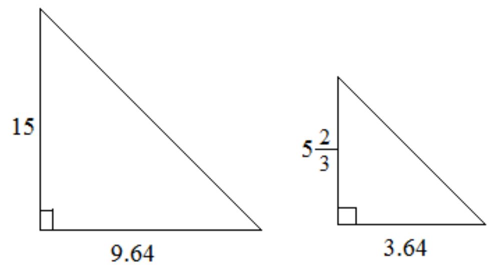
$$\frac{15}{6 + X} = \frac{5\frac{2}{3}}{X}$$

$$15X = 34 + 5\frac{2}{3}X$$

$$9\frac{1}{3}X = 34$$

$$X \cong 3.64 \text{ feet}$$

Step 4: Check answer



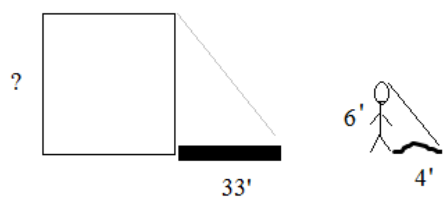
$$\frac{15}{9.64} = 1.56 \checkmark$$

$$\frac{5.6\overline{6}}{3.64} = 1.56 \checkmark$$

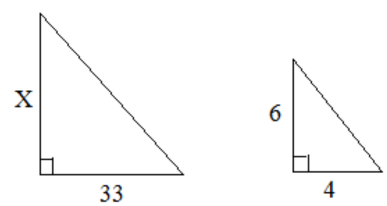
Example: A 6-foot tall man casts a shadow 4 feet. And, a house casts a shadow 33 feet. What is the height of the house?

Solution:

Step 1: Draw a picture



Step 2: Split into triangles



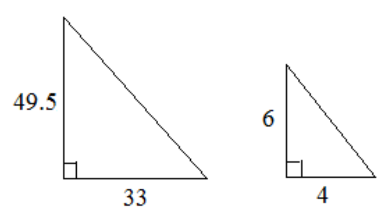
Step 3: Solve proportion

$$\frac{X}{33} = \frac{6}{4}$$

(cross multiply) $4X = 6(33)$

$$X = 49.5 \text{ feet}$$

Step 4: Check answer

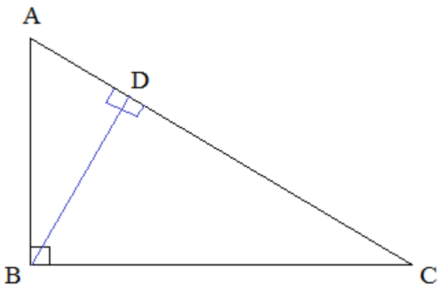


$$\frac{49.5}{6} = 8.25 \checkmark$$

$$\frac{33}{4} = 8.25 \checkmark$$

Altitude and 3 similar right triangles

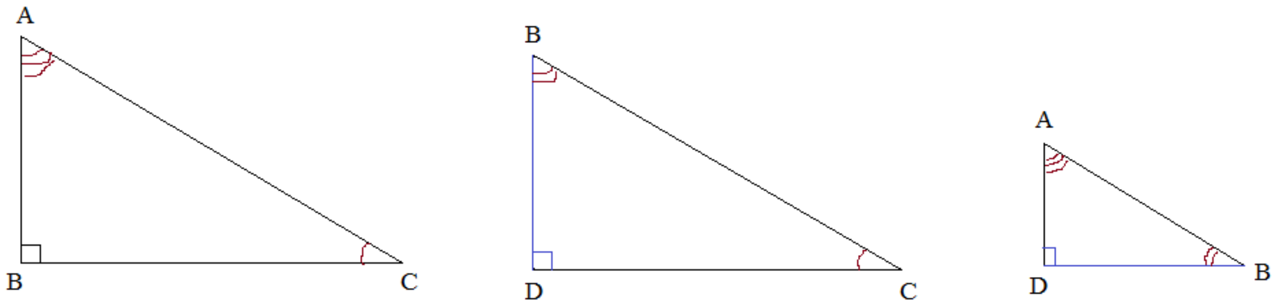
An altitude of a right triangle, extending from the right angle vertex to the hypotenuse, creates 3 similar triangles!



$\overline{BD}$ is an altitude extending from vertex B to $\overline{AC}$

($\overline{AB}$ and $\overline{BC}$ are the other altitudes of the triangle)

Then, displaying the 3 right triangles (facing the same direction), we can observe the congruent parts and the similarity:



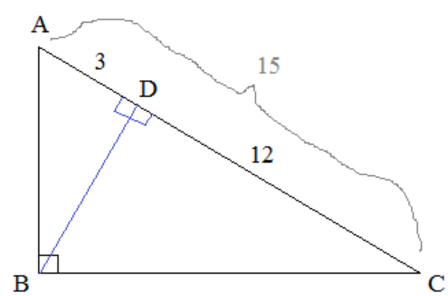
Using Angle-Angle Theorem of similarity proves all 3 triangles are similar (and proportional) to each other....

Example: Given: right triangle ABC with altitude BD

$\overline{AD} = 3$

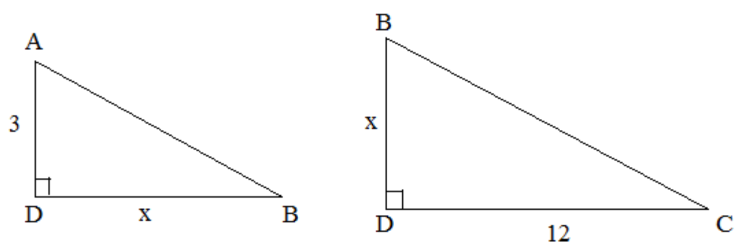
$\overline{AC} = 15$

Find the length of the altitude $\overline{BD}$.



Step 1: Draw the figure; label parts

Step 2: Separate the triangles; set up ratios/proportions



$$\frac{3}{x} = \frac{x}{12}$$

Step 3: Solve

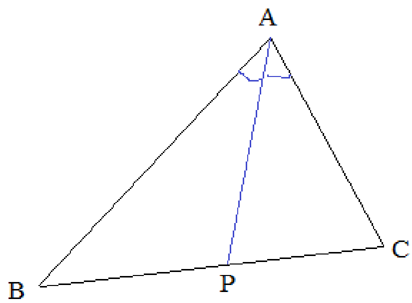
$$x^2 = 36$$

$$x = 6$$

The altitude is 6 units...
Also, the altitude is the *geometric mean* of the 2 segments that form the hypotenuse!

Angle Bisector Theorem

Definition: The angle bisector divides the opposite side into 2 parts with the same relative lengths (ratio) as the other two sides of the triangle.



$\overline{AP}$ is the angle bisector

Opposite side is divided into two parts: $\overline{BP}$ $\overline{CP}$

Ratio of other two sides of triangle: $\dfrac{\overline{AC}}{\overline{AB}}$

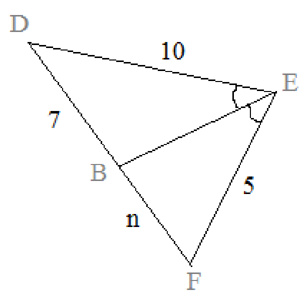
$$\frac{\overline{AC}}{\overline{AB}} = \frac{\overline{CP}}{\overline{BP}}$$

Also, simple algebra can show that ratios of "new triangle sides" are the same!

$$\frac{AC}{AB} = \frac{CP}{BP} \longrightarrow AC \cdot BP = AB \cdot CP \longrightarrow \frac{AC \cdot BP}{CP} \cdot AB \longrightarrow \frac{\overline{AC}}{\overline{CP}} = \frac{\overline{AB}}{\overline{BP}}$$

Example:

What is n?



Since $\angle DEB \cong \angle FEB$, $\overline{BE}$ is an angle bisector...

Therefore, we can use the angle bisector theorem to find n:

$$\frac{DE}{EF} = \frac{DB}{BF} \qquad \text{or,} \qquad \frac{DE}{DB} = \frac{EF}{BF}$$

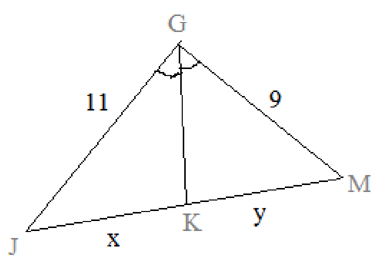
$$\frac{10}{5} = \frac{7}{n} \qquad\qquad \frac{10}{7} = \frac{5}{n}$$

$n = 3.5$ $\qquad$ $n = 3.5$

Example: Find x and y:

$x + y = 14$

$\angle GKJ \cong \angle GKM$



Using the angle bisector theorem and substitution:

$$\frac{11}{x} = \frac{9}{(14 - x)}$$

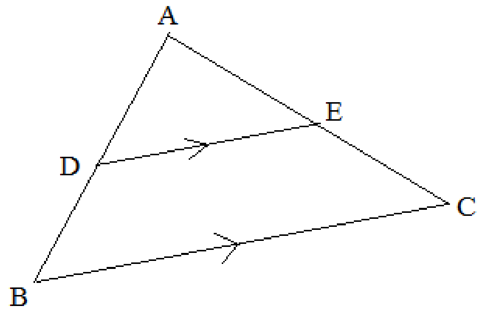
$9x = 154 - 11x$

$20x = 154$

$x = 7.7$ then, $y = 6.3$

Side-Splitter Theorem

Definition: If a line is parallel to one side of a triangle, then it splits the other two sides proportionally.



$\overline{DE}$ is parallel to $\overline{BC}$

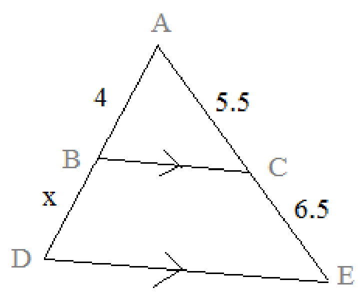
$\overline{DE}$ splits triangle ABC

$$\frac{\overline{AD}}{\overline{DB}} = \frac{\overline{AE}}{\overline{EC}}$$

Also, simple algebra can show that the ratio of the "upper parts" is the same as the ratio of the "lower parts".

$$\frac{AD}{DB} = \frac{AE}{EC} \longrightarrow AD \cdot EC = DB \cdot AE \longrightarrow \frac{AD \cdot EC}{AE} = DB \longrightarrow \frac{\overline{AD}}{\overline{AE}} = \frac{\overline{DB}}{\overline{EC}}$$

Example: Find x:



Since BC || DE, we can use side-splitter to find x.

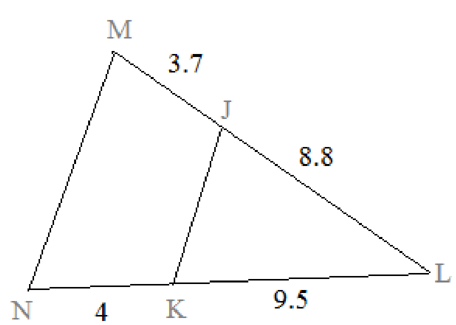
$$\frac{AB}{BD} = \frac{AC}{CE}$$

$$\frac{4}{x} = \frac{5.5}{6.5}$$

$$5.5x = 26$$

$x = 4.\overline{72}$

Example: Is $\overline{MN}$ parallel to $\overline{JK}$?



Compare the ratios/proportions:

$$\frac{JL}{JM} = \frac{8.8}{3.7} = 2.\overline{378}$$

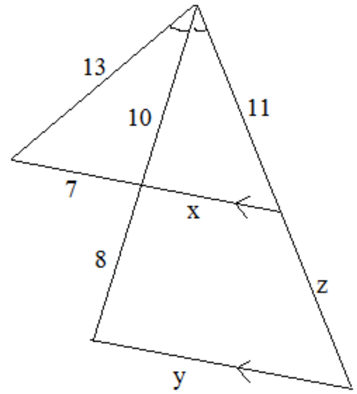
$$\frac{KL}{KN} = \frac{9.5}{4} = 2.375$$

The proportions are NOT equal; therefore, JK is not parallel to MN!!

Similar Triangles, Angle Bisector Theorem, & Side-Splitter

Example: Given the labeled diagram,

Find x, y, and z



Find x: (angle bisector theorem)

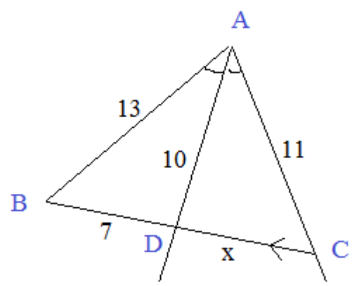
($\overline{AD}$ bisects angle A)

$$\frac{AB}{BD} = \frac{AC}{DC}$$

$$\frac{13}{7} = \frac{11}{x}$$

$$13x = 77$$

$$x = 5\frac{12}{13}$$



Find y: (similar triangles)

Since DC || EF, $\angle D = \angle E$ $\angle C = \angle F$

(parallel lines cut by transversals)

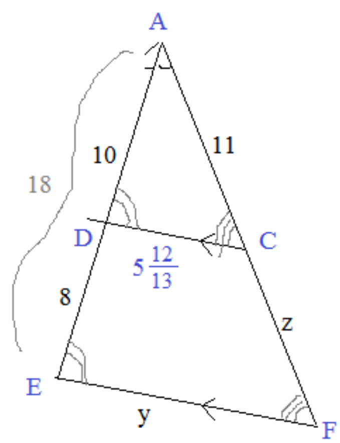
$\triangle ADC \sim \triangle AEF$ (Angle-Angle similarity theorem)

$$\frac{AD}{AE} = \frac{DC}{EF}$$

$$\frac{10}{18} = \frac{5\frac{12}{13}}{y}$$

$$10y \cong 106.6$$

$$y \cong 10.66$$



Find z: (side-splitter theorem)

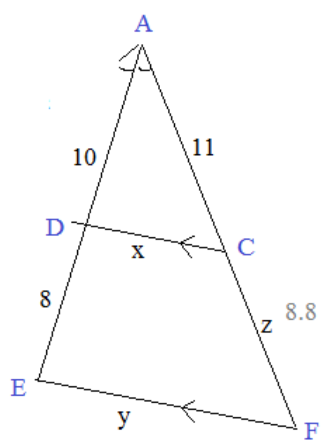
DC || EF , so use side-splitter...

$$\frac{AD}{DE} = \frac{AC}{CF}$$

$$\frac{10}{8} = \frac{11}{z}$$

$$10z = 88$$

$$z = 8.8$$



Note: using similar triangles, we can check z = 8.8

$$\frac{AD}{AE} = \frac{10}{18} = .5556 \checkmark$$

$$\frac{AC}{AF} = \frac{11}{19.8} = .5556 \checkmark$$

Triangles, Proportions, and Geometric Mean

Arithmetic Mean between two numbers: the average, midpoint, or middle.

Example: What is the (arithmetic) mean of 6 and 10?

To find the average: $\frac{6 + 10}{2} = 8$

8 is the midpoint: two units from 6 and two units from 10 (a common difference of 2)

Geometric Mean between two numbers: a middle where there is a common ratio between the numbers

Example: What is the geometric mean of 4 and 16?

To find the geometric mean, *multiply the 2 numbers and then take the square root*.

$4 \times 16 = 64 \qquad \sqrt{64} = 8$

8 is the middle: 4 8 16

x 2 x 2

(a *common ratio* of 2)

"Means/Extremes"

Examples: $\frac{3}{7} = \frac{12}{28}$ Means are 7 and 12; Extremes are 3 and 28...

3:4 = 12:16 Means are 4 and 12; Extremes are 3 and 16...

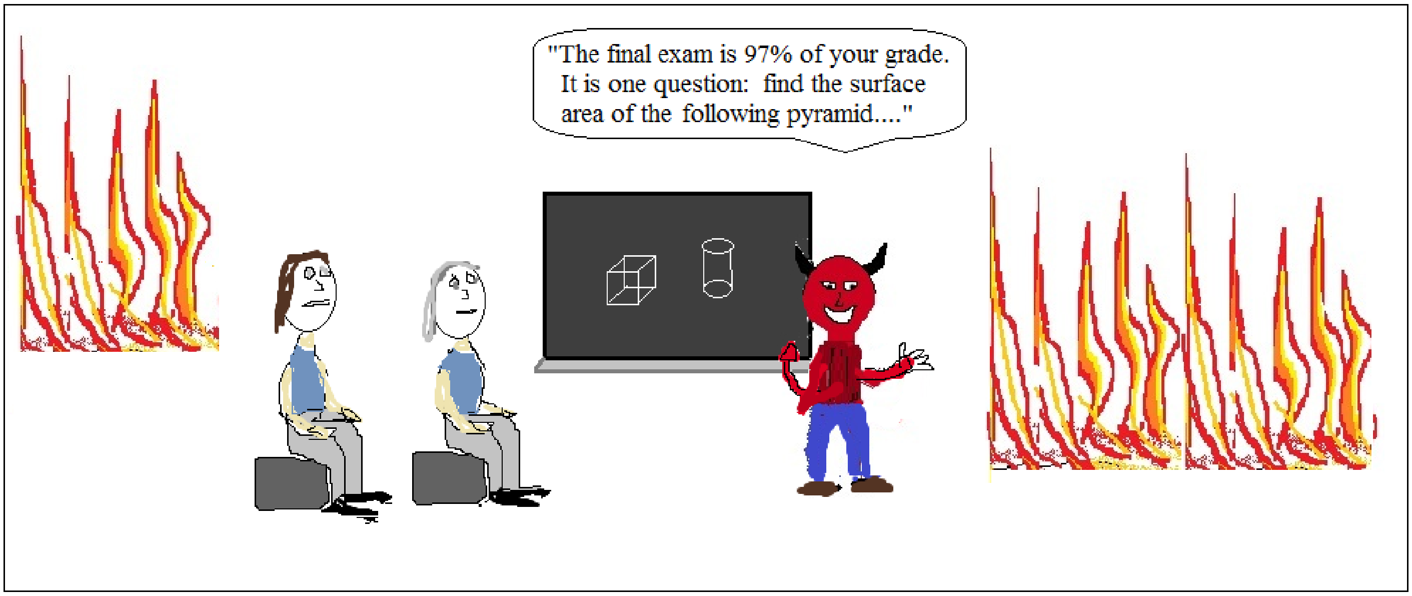
The geometric mean can be found in ratios/proportions..

Examples: 1:3 = 3:9 *Means* -- 3 (middle) *Extremes* = 1 and 9 (outer)

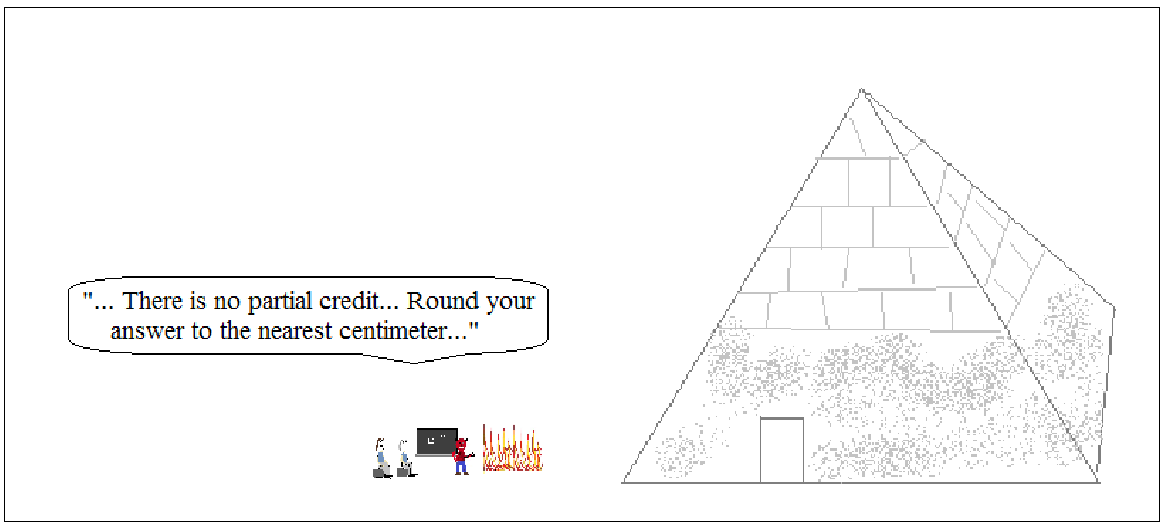
The means are both 3; therefore, 3 is the geometric mean of 1 and 9

$\frac{4}{8} = \frac{8}{16}$ The mean is 8, and the extremes are 4 and 16...

The *geometric mean* of 4 and 16 is 8



Math in Hell



In its 1000 year history, no one ever passed Mr. Devlin's Geometry class.

LanceAF #39 7-1-12
www.mathplane.com

Practice Exercises -→

Similar Triangles, Ratios, and Geometric Mean

I. Means

1) What is the arithmetic mean of 6 and 96?

2) What is the geometric mean of 6 and 96?

3) Given: $k = 4$ $m = 20$

   a) If m is the arithmetic mean of k and p, what is p?

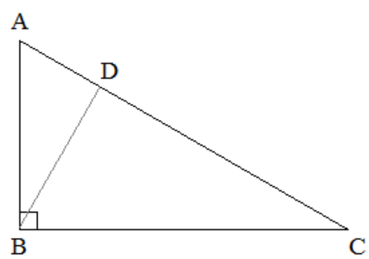
   b) If m is the geometric mean between k and r, what is r?

II. Ratios and Proportions

1) 10:x and x:20 have the same ratios. What is x?

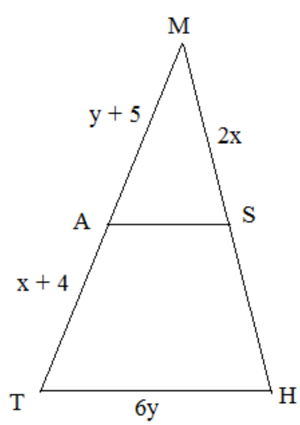
2) $\frac{3}{12} = \frac{x}{60}$

3) The hypotenuse of triangle ABC is 20 units; If $\overline{AD}$ is 4 units, what is the length of altitude $\overline{BD}$?



4) Given: $\overline{MT} \cong \overline{MH}$
   $\overline{AS}$ bisects $\overline{MH}$
   $\overline{AS}$ bisects $\overline{MT}$

   What is the perimeter of $\triangle MTH$?

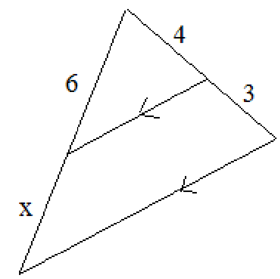


III. Similar triangles and Theorems

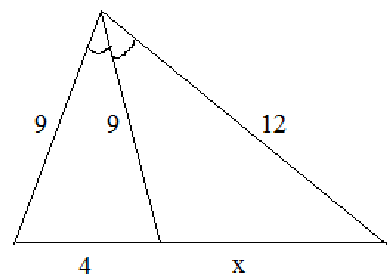
Similar Triangles, Ratios, and Geometric Mean

Find x:

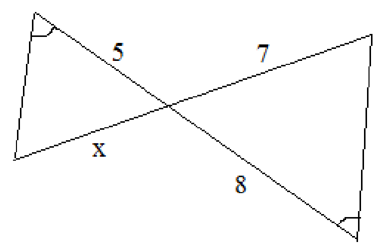
1)



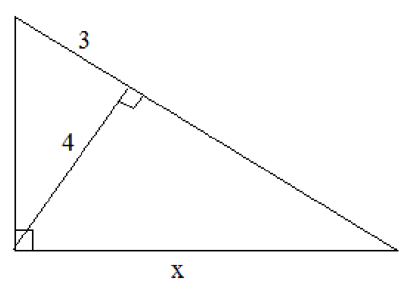
2)



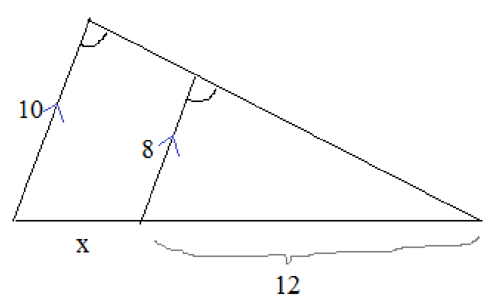
3)



4)



5)

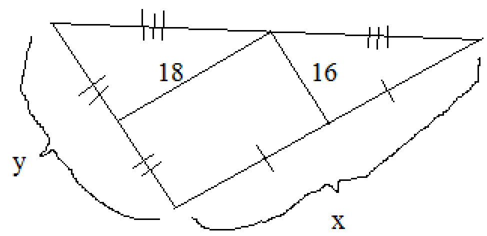


IV. Midsegments and More...

Similar Triangles, Ratios, and Geometric Mean

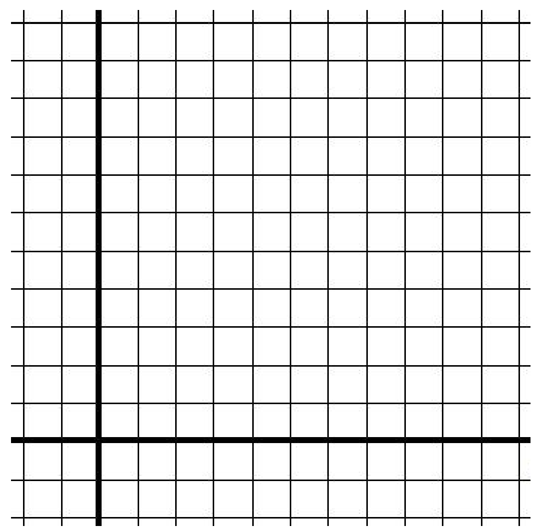
1) What is x?

What is y?



2) The coordinates of the vertices of a triangle are A (1, 3) B (5, 7) C (3, -1)
If H is the midpoint of $\overline{AC}$ and J is the midpoint of $\overline{BC}$,

a) Find H; Find J



b) Graph the triangle, and label the points

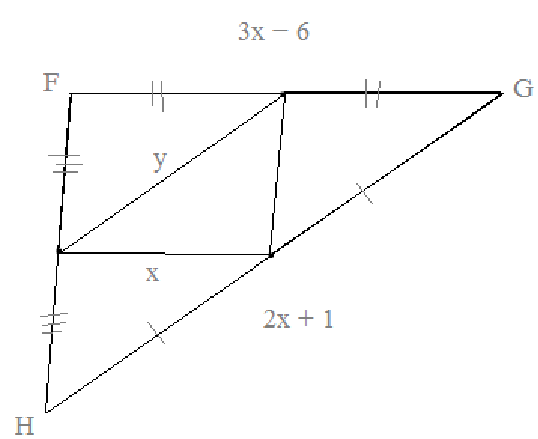
c) Verify (algebraically) that $\overline{AB}$ is parallel to $\overline{HJ}$

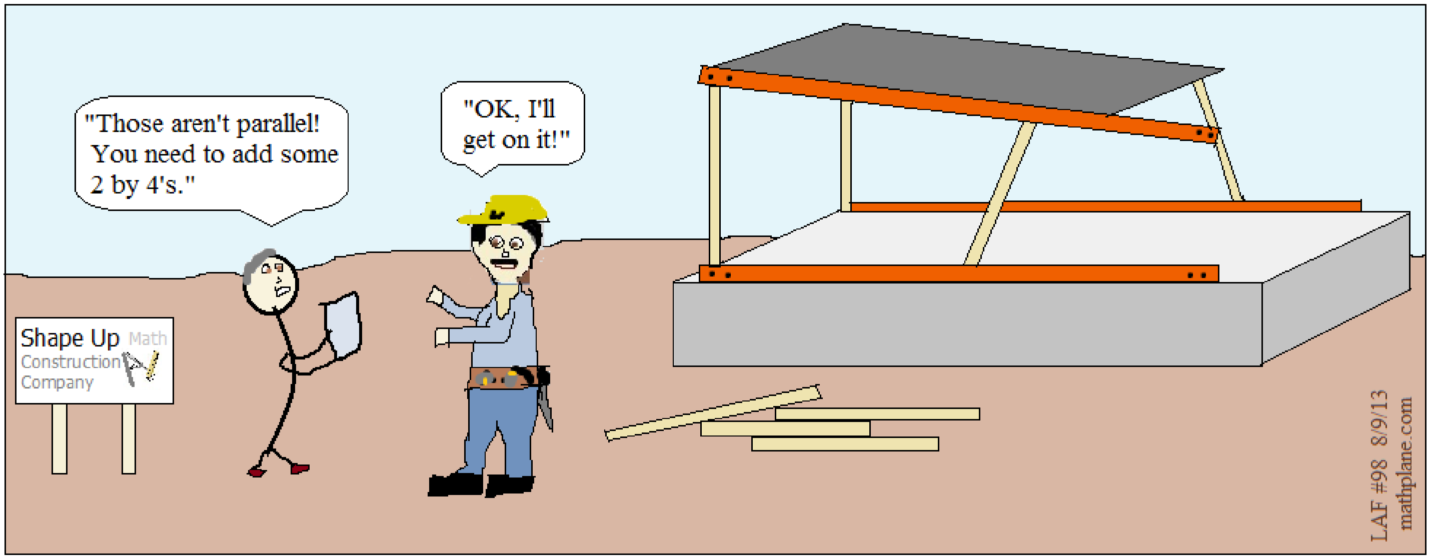
d) Verify $\overline{HJ} = (1/2)\overline{AB}$

3) Find x and y

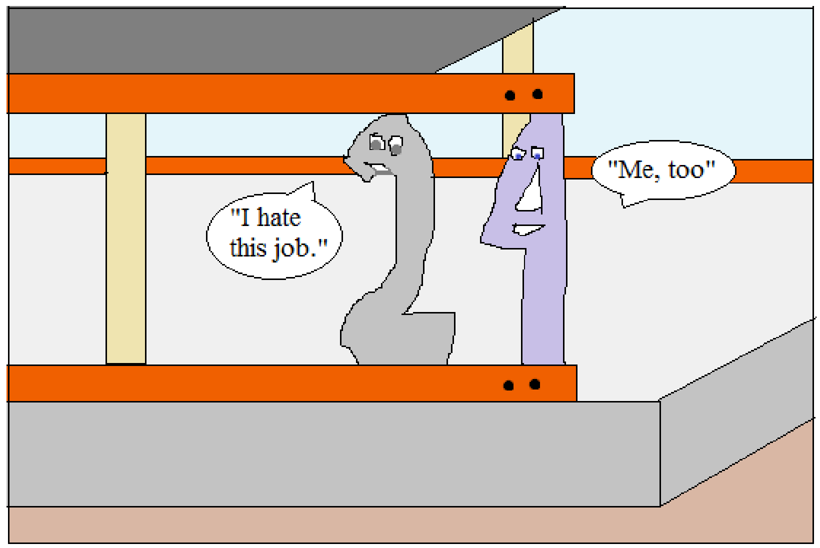
FG = 3x − 6

GH = 2x + 1





The Math Guy misunderstood the Architect's suggestion...



Building Materials

SOLUTIONS -→

Similar Triangles, Ratios, and Geometric Mean

Solutions

I. Means

1) What is the arithmetic mean of 6 and 96?

$\frac{6+96}{2} = \boxed{51}$

6 51 96
+45 +45

2) What is the geometric mean of 6 and 96?

$\sqrt{6 \times 96} = \boxed{24}$

6 24 96
x 4 x 4

3) Given: k = 4 m = 20

a) If m is the arithmetic mean of k and p, what is p?

k m p
4 20 p
+16 +16

$\boxed{p = 36}$

$\frac{k+p}{2} = m$

$4 + p = 40$

b) If m is the geometric mean between k and r, what is r?

k m r
4 20 r
x 5 x 5

$\boxed{r = 100}$

$\sqrt{kr} = m$

$\sqrt{4r} = 20$

II. Ratios and Proportions

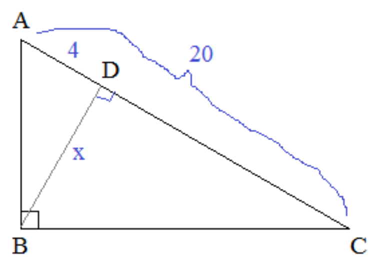
1) 10:x and x:20 have the same ratios. What is x?

$\frac{10}{x} = \frac{x}{20}$ $x^2 = 200$ $\boxed{x = 10\sqrt{2} \text{ or } -10\sqrt{2}}$

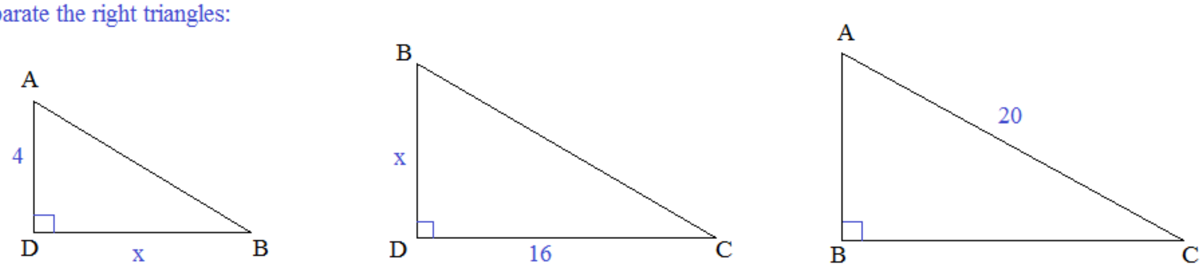
2) $\frac{3}{12} = \frac{x}{60}$

$12x = 180$ $\boxed{x = 15}$

3) The hypotenuse of triangle ABC is 20 units; If $\overline{AD}$ is 4 units, what is the length of altitude $\overline{BD}$?



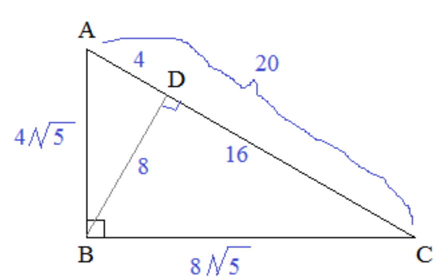
Separate the right triangles:



Use the first 2 triangles and ratios:

$\frac{4}{x} = \frac{x}{16}$ $x^2 = 64$ $\boxed{x = 8}$

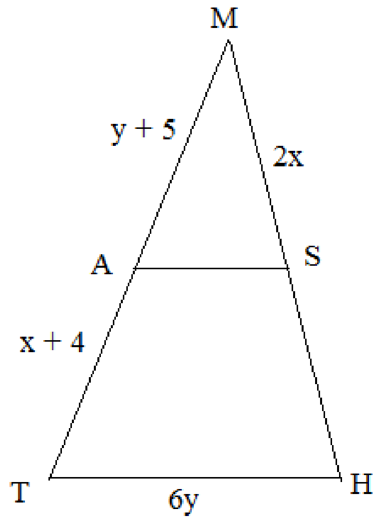
Check answers and other sides:



Solution

4) Given: $\overline{MT} \cong \overline{MH}$
$\overline{AS}$ bisects $\overline{MH}$
$\overline{AS}$ bisects $\overline{MT}$

What is the perimeter of $\triangle MTH$?



$y + 5 = x + 4$
(because A is the midpoint of MT)

$2x = y + 5$
(because S is midpoint of MH and MT = MH)

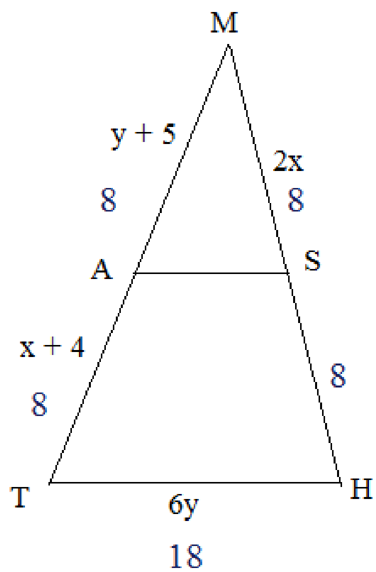
Since we have 2 equations and 2 unknowns, we can solve:

(rewrite equations)

$x - y = 1$
$2x - y = 5$

(elimination method)

$-x = -4$
$x = 4$
$y = 3$



Substitute into the triangle,
add up the segments:
$8 + 8 + 18 + 8 + 8 = \boxed{50}$

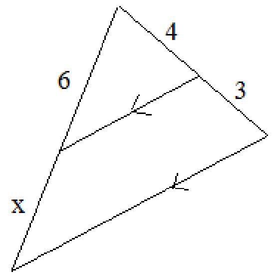
III. Similar triangles and Theorems

Solutions

Similar Triangles, Ratios, and Geometric Mean

Find x:

1)



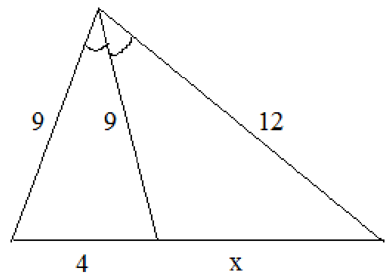
Use Side-splitter theorem:

$$\frac{6}{x} = \frac{4}{3}$$

$4x = 18$

$x = 4.5$

2)



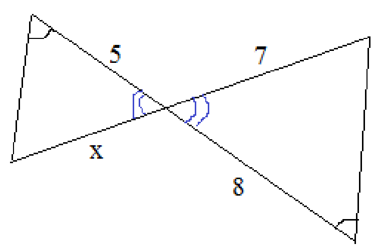
Use Angle bisector theorem:

$$\frac{9}{4} = \frac{12}{x}$$

$9x = 48$

$x = 5\frac{1}{3}$

3)



Use Angle-Angle theorem and similar triangles:

$$\frac{5}{8} = \frac{x}{7}$$

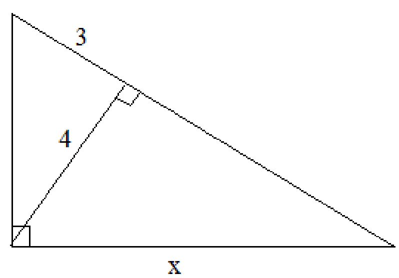
$8x = 35$

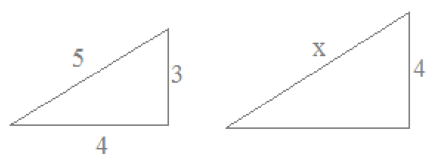
$x = 4.375$

4)

pythagorean triplet 3-4-5

5

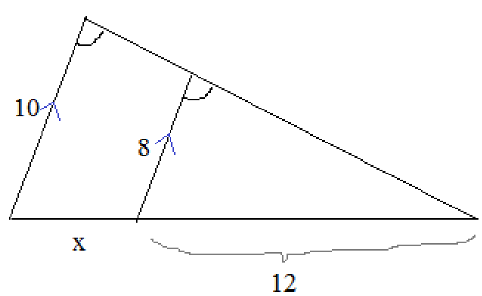




$$\frac{5}{x} = \frac{3}{4}$$

$x = 6\frac{2}{3}$

5)



Since corresponding angles are congruent, the line segments must be parallel...

$$\frac{8}{10} = \frac{12}{x + 12}$$

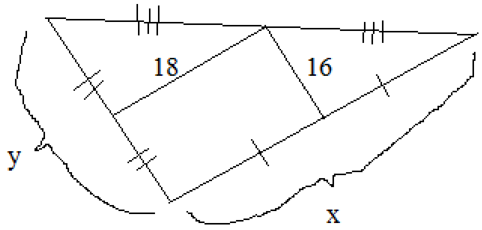
$x + 12 = 15$

$x = 3$

IV. Midsegments and More... SOLUTIONS Similar Triangles, Ratios, and Geometric Mean

1) What is x? 36 (18 x 2)

What is y? 32 (16 x 2)



2) The coordinates of the vertices of a triangle are A (1, 3) B (5, 7) C (3, -1)
If H is the midpoint of $\overline{AC}$ and J is the midpoint of $\overline{BC}$,

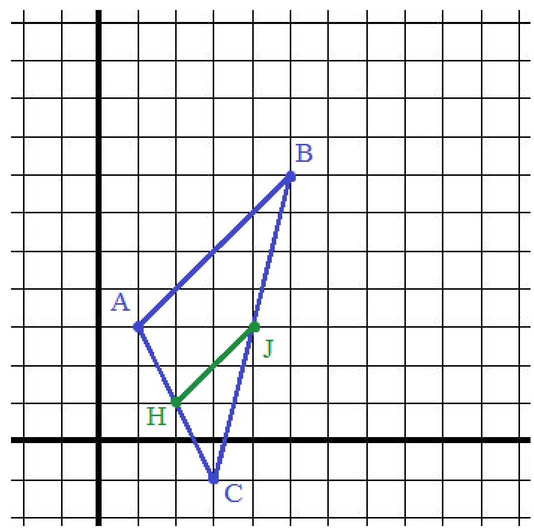
a) Find H; Find J

midpoint formula $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

midpoint of A and C: H (2, 1)

midpoint of B and C: J (4, 3)

b) Graph the triangle, and label the points



c) Verify (algebraically) that $\overline{AB}$ is parallel to $\overline{HJ}$

If 2 segments are | |, then slopes are =

slope of $\overline{AB}$: $\frac{7-3}{5-1} = 1$ ✓

slope of $\overline{HJ}$: $\frac{4-2}{3-1} = 1$ ✓

d) Verify $\overline{HJ} = (1/2)\overline{AB}$

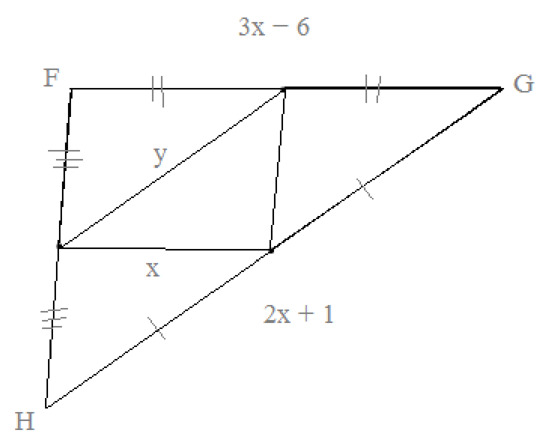
since HJ | | AB and H and J are midpoints, HJ = (1/2)AB

using distance formula:

$\overline{HJ} = \sqrt{(4-2)^2 + (3-1)^2} = \sqrt{8} = 2\sqrt{2}$ ✓

$\overline{AB} = \sqrt{(1-5)^2 + (3-7)^2} = \sqrt{32} = 4\sqrt{2}$ ✓

3) Find x and y



Using midsegment/side-splitter theorems,

FG = 3x − 6

GH = 2x + 1

we know $x = (1/2)(3x - 6)$

$x = \frac{3x}{2} - 3$

$\frac{-x}{2} = -3$

$x = 6$

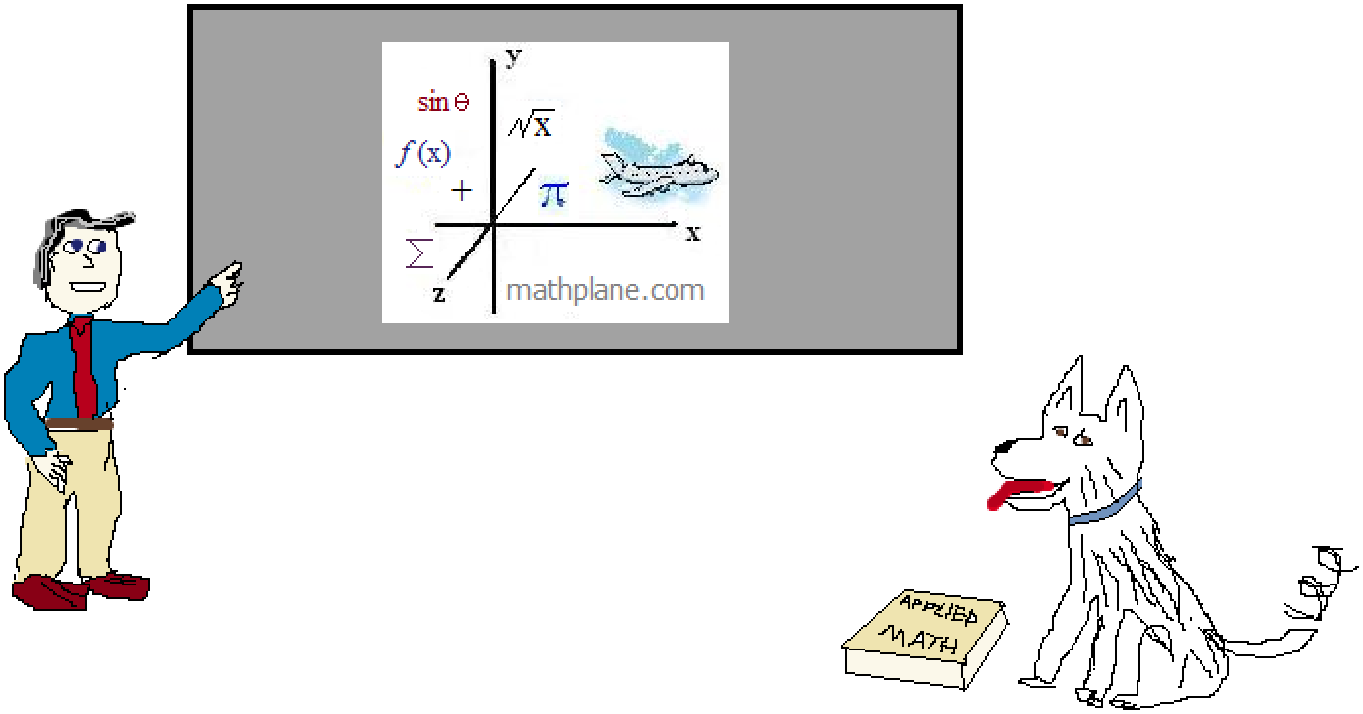
since x = 6,

HG = 2(6) + 1 = 13

therefore, y = 13/2

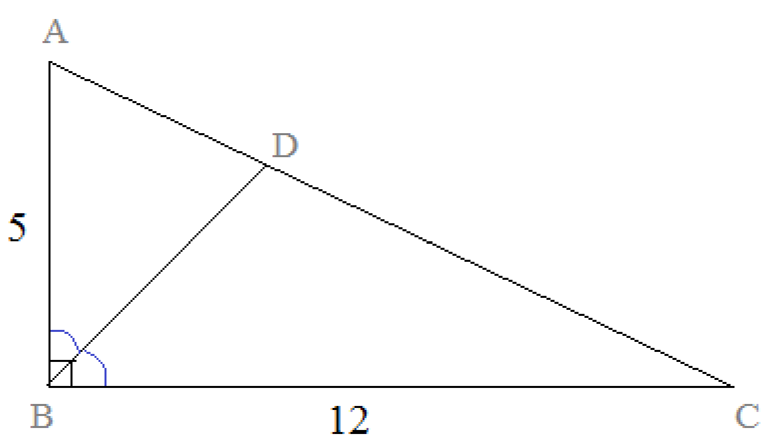
Thanks for visiting. (Hope it helped!)

If you have suggestions, questions, or requests, let us know.   Enjoy!



ONE MORE QUESTION:

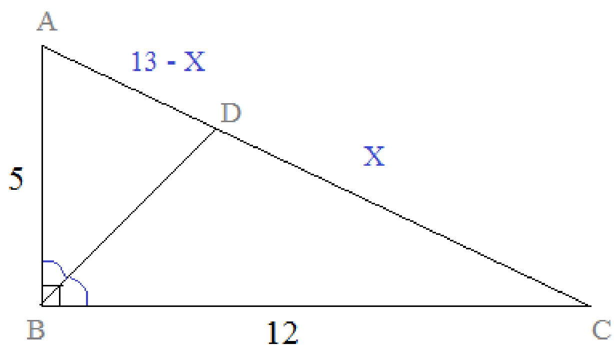
Find the length of $\overline{CD}$:



ABC is a right triangle, where $\angle ABD \cong \angle CBD$

(ANSWER on next page)

Find the length of $\overline{CD}$:



ABC is a right triangle, where $\angle ABD \cong \angle CBD$

Answer:

Since ABC is a right triangle with legs 5 and 12, we know the hypotenuse is 13...

If $\overline{CD} = X$, then $\overline{AD} = 13 - X$

Since $\overline{BD}$ is a bisector, we can use the (triangle) angle bisector theorem

$$\frac{AB}{BC} = \frac{AD}{DC}$$

$$\frac{5}{12} = \frac{13 - X}{X}$$

$$5X = 156 - 12X$$

$$17X = 156$$

$$\boxed{X \approx 9.176}$$