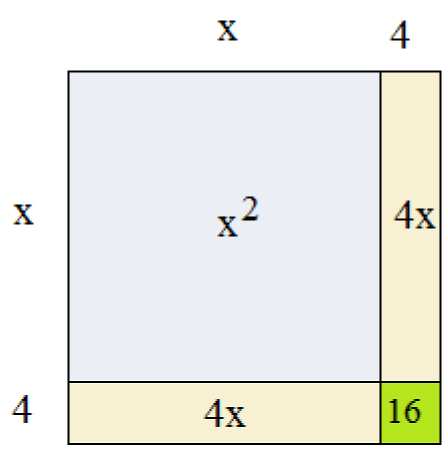
Completing the Square & the Quadratic Formula

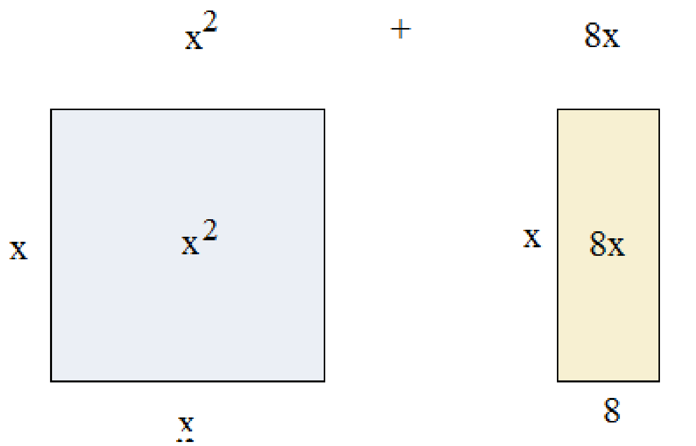
Notes, Examples, and Practice Exercises (with Solutions)

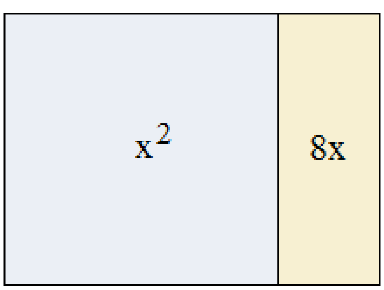


$x^2 + 8x + 16$

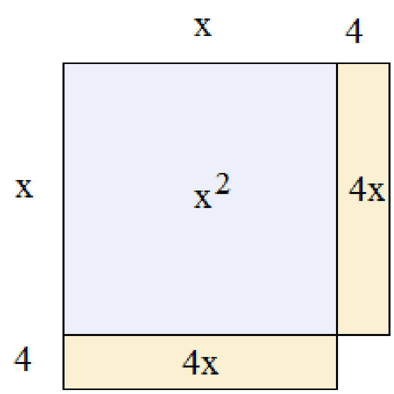
Topics include discriminant, geometric display, standard form of a circle, deriving the quadratic formula, maximum of a parabola, and more.

Completing the Square (Geometrically)

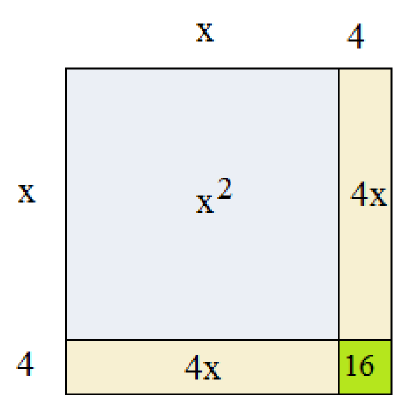




$x^2 + 8x$



What is missing? A square of measure 4 x 4



$x^2 + 8x + 16$

Completing the Square

What is it? A technique for simplifying or solving quadratic equations.

Example: Solve $x^2 + 3x - 11 = 0$

Comments/Notes

Step 1: Separate the variables (x)

$$x^2 + 3x \quad = 11$$

The coefficient of the first term must be 1.

Step 2: "Complete the square" using

$$\left(\frac{b}{2}\right)^2$$

The term is added to both sides so that the equation does not change.

$$x^2 + 3x + \frac{9}{4} = 11 + \frac{9}{4}$$

Step 3: Factor

$$(x + \frac{3}{2})(x + \frac{3}{2}) = \frac{53}{4}$$

$$(x + \frac{3}{2})^2 = \frac{53}{4}$$

The factored trinomial becomes a perfect square!

Step 4: Solve

$$\sqrt{(x + \frac{3}{2})^2} = \sqrt{\frac{53}{4}}$$

$$(x + \frac{3}{2}) = \pm\frac{\sqrt{53}}{2}$$

Taking the square root of a square results in + or − solution

$$\boxed{x = \frac{-3}{2} \pm \frac{\sqrt{53}}{2}}$$

Example: Change the following quadratic equation into *vertex form*.

$y = 2x^2 + x - 28$ What is the vertex?

Step 1: Separate the variables

$$2x^2 + x \qquad - 28$$

Step 1a: Change lead coefficient to 1 1

$$2(x^2 + \frac{x}{2} \qquad) - 28$$

Step 2: "Complete the square"

$$b = 1/2 \quad \text{so,} \quad \left(\frac{b}{2}\right)^2 = \frac{1}{16}$$

$$2(x^2 + \frac{x}{2} + \frac{1}{16}) - 28 - \frac{2}{16}$$

NOTE: we add $2 x \frac{1}{16}$ to complete the square; then, we subtract the same quantity so the equation remains unchanged

Step 3: Factor and simplify

$$2(x + \frac{1}{4})^2 - \frac{450}{16}$$

$$\boxed{2(x + \frac{1}{4})^2 - \frac{225}{8}}$$

Step 4: Solve

Vertex form: $y = a(x - h)^2 + k$

Vertex is (h, k):

$$\boxed{\left(-\frac{1}{4}, \frac{-225}{8}\right)}$$

Example: The following is the general equation of a circle. What is the center of the circle?

$$x^2 + 10x + y^2 - 8y + 32 = 0$$

Change to standard form of a circle:

$$(x - h)^2 + (y - k)^2 = r^2$$

where (h, k) is the center and r is the radius

Complete the squares to answer:

Step 1: Separate the variables

$$x^2 + 10x + \quad y^2 - 8y + \quad 32 = 0$$

$$x^2 + 10x + \quad y^2 - 8y + \quad = -32$$

Step 2: Complete the squares

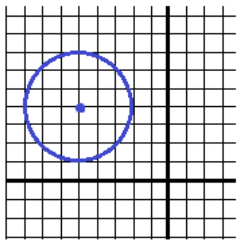
$$x^2 + 10x + 25 + y^2 - 8y + 16 = -32 + 25 + 16$$

Use $\left(\frac{b}{2}\right)^2$ Add to left side... .... must add to right side

Step 3: Factor the (perfect square) trinomials

$$(x + 5)(x + 5) + (y - 4)(y - 4) = 9$$

$$(x + 5)^2 + (y - 4)^2 = 9$$



Step 4: Answer

$(x - h)^2 + (y - k)^2 = r^2$ Then, h = -5 k = 4 Center is $(-5, 4)$
Radius is 3

points include: (-5, 1) (-5, 7) (-2, 4) (-8, 4)

Example: What is the maximum of this function?

$$3x^2 - 18x + y + 22 = 0$$

This is the general form of a parabola. If we complete the square, we will reveal the vertex (maximum, because this parabola faces down)

Step 1: Rearrange and separate the variables

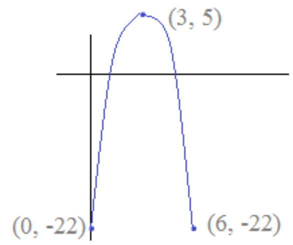
$$y + 22 = -3x^2 + 18x$$

Change to vertex form of a parabola:

$$y = a(x + h)^2 + k$$

Step 1a: Change lead coefficient to 1

$$y + 22 = -3(x^2 - 6x \quad)$$



Step 2: Complete the square

$$y + 22 + -3(9) = -3(x^2 - 6x + 9)$$

Step 3: Factor the trinomial and simplify

$$y - 5 = -3(x - 3)(x - 3)$$

$$y = -3(x - 3)^2 + 5$$

h = 3 k = 5 the vertex is (3, 5) which is the maximum of this function

Example: Graph the function $f(x) = x^2 - 2x$

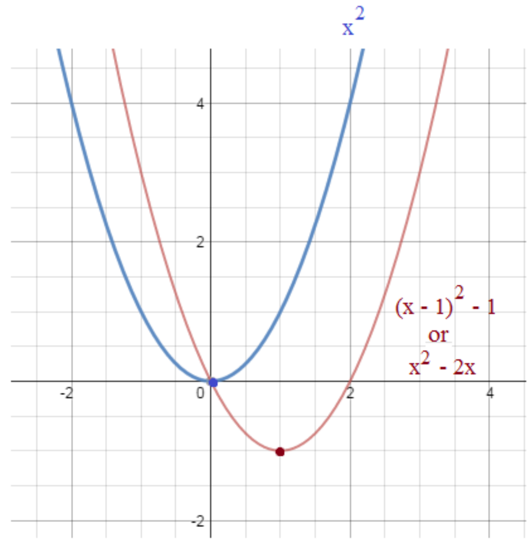
This quadratic does not have a "direct parent function"....
But, if we complete the square:

$$x^2 - 2x + 1 \longrightarrow (x-1)(x-1) = (x-1)^2$$

Then, compare the result with the original function:

$$x^2 - 2x + 1 = (x-1)^2$$

so, $x^2 - 2x = \boxed{(x-1)^2 - 1}$



Now, let's graph:

parent function: x^2

horizontal shift: 1 unit to the right

vertical shift: 1 unit down

Example: Graph the function $x^2 + 4x + 7$
(by completing the square and using the parent function)

Take the <u>quadratic term and linear term,</u> $x^2 + 4x$, and complete the square

$$x^2 + 4x + 4 \longrightarrow (x+2)(x+2) = (x+2)^2$$

$$x^2 + 4x + 4 = (x+2)^2$$

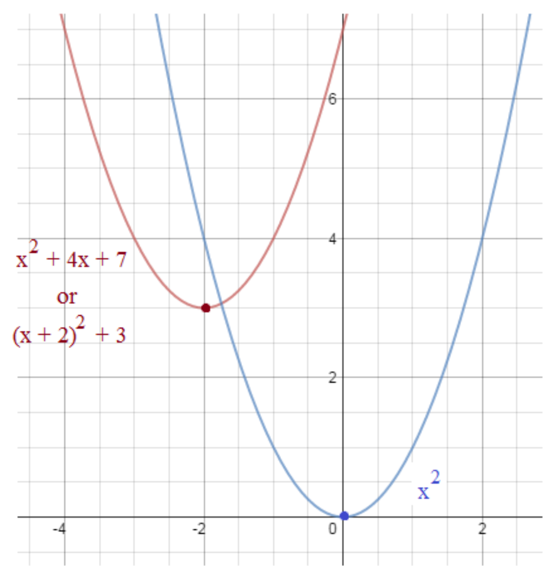
so, $x^2 + 4x + 7 = \boxed{(x+2)^2 + 3}$

Now, let's graph:

parent function: x^2

horizontal shift: 2 units to the left

vertical shift: 3 units up



Factor by completing the square (advanced)

Example: $x^4 + 2x^2 + 9$

create a perfect square trinomial

$$x^4 + 2x^2 + 4x^2 + 9 - 4x^2$$

$$x^4 + 6x^2 + 9 - 4x^2$$

factor

$$(x^2 + 3)^2 - 4x^2 \quad \text{(difference of squares)}$$

$$[(x^2 + 3) + 2x][(x^2 + 3) - 2x]$$

$$(x^2 + 2x + 3)(x^2 - 2x + 3)$$

Example: $x^4 + 4$

create a perfect square trinomial

$$x^4 + 4x^2 + 4 - 4x^2$$

$$x^4 + 4x^2 + 4 - 4x^2$$

$$(x^2 + 2)(x^2 + 2) - 4x^2$$

$$(x^2 + 2)^2 - 4x^2 \quad \text{(difference of squares)}$$

$$(x^2 + 2 + 2x)(x^2 + 2 - 2x)$$

Example: $x^4 - 18x^2 + 1$

(create a perfect square trinomial by splitting the middle)

$$x^4 - 2x^2 + 1 - 16x^2$$

$$(x^2 - 1)(x^2 - 1) - 16x^2$$

$$(x^2 - 1)^2 - 16x^2$$

$$[(x^2 - 1) + 4x][(x^2 - 1) - 4x]$$

$$(x^2 + 4x - 1)(x^2 - 4x - 1)$$

Example: Solve $x^4 - 19x^2 + 25 = 0$

$$x^4 - 10x^2 + 25 - 9x^2$$

$$(x^2 - 5)^2 - 9x^2$$

$$[(x^2 - 5) + 3x][(x^2 - 5) - 3x]$$

$$(x^2 + 3x - 5)(x^2 - 3x - 5) = 0$$

$$(x^2 + 3x - 5) = 0 \qquad (x^2 - 3x - 5) = 0$$

quadratic formula

$$x = \frac{-3 \pm \sqrt{9 + 20}}{2} \qquad x = \frac{3 \pm \sqrt{9 + 20}}{2}$$

Quadratic Formula

The quadratic formula is derived from 'completing the square'.

It can be used to find the roots of a quadratic equation
(i.e. "what values of x equal zero")

So, it can be used to factor a quadratic equation.

Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Examples:

1) Solve using the quadratic formula

$3x^2 + 2x - 5 = 0$

a = 3
b = 2
c = -5

$$x = \frac{-2 \pm \sqrt{2^2 + 4(3)(-5)}}{2(3)} = \frac{-2 \pm \sqrt{64}}{6}$$

$$\frac{-2+8}{6} = 1 \qquad \frac{-2-8}{6} = \frac{-5}{3}$$

$x = \frac{-5}{3}, 1$

2) Factor the following function

$x^2 + 10x + 21$

a = 1
b = 10
c = 21

$$\frac{-10 \pm \sqrt{10^2 - 4(1)(21)}}{2(1)} = \frac{-10 \pm \sqrt{16}}{2}$$

$$\frac{-10+4}{2} = -3 \qquad \frac{-10-4}{2} = -7$$

-3 and -7 are *zeros* of the quadratic..

Therefore, (x + 3) and (x + 7) are *factors*.

$x^2 + 10x + 21 = (x + 3)(x + 7)$

The Discriminant:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$b^2 - 4ac$ is the discriminant

It reveals the type of roots that a quadratic has.

$b^2 - 4ac > 0$ then 2 real roots
$b^2 - 4ac = 0$ then 1 real root
$b^2 - 4ac < 0$ then 0 real roots

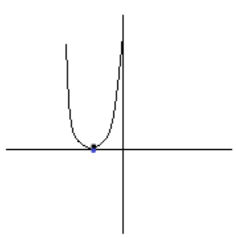
Examples:

$x^2 + 8x + 16$

discriminant is $b^2 - 4ac$

$= 8^2 - 4(1)(16) = 0$

one x-intercept

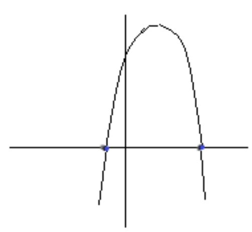


$-x^2 + 5x + 14$

discriminant is $b^2 - 4ac$

$= 5^2 - 4(-1)(14) = 81 > 0$

two x-intercepts (zeros)



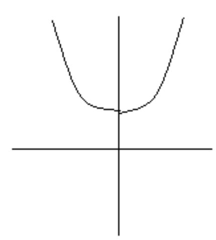
$x^2 + 4$

a = 1
b = 0
c = 4

discriminant is $-16 < 0$

there are no *real* roots

(2 imaginary roots: $2i$ and $-2i$)



Completing the Square: *Deriving the Quadratic Formula*

Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$ax^2 + bx + c = 0$

Solve for x (by completing the square):

separate the variable: $ax^2 + bx \quad + c = 0$

change lead coefficient to 1 (factor out the 'a'): $a(x^2 + \frac{b}{a}x \quad) \quad + c = 0$

complete the square by adding $\left(\frac{\frac{b}{a}}{2}\right)^2$: $a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) \quad + c = 0 + \frac{b^2}{4a}$

Since we added $a\left(\frac{b}{4a^2}\right)$ on the left side, we add $\frac{b^2}{4a}$ to the right side...

$$a(x + \frac{b}{2a})(x + \frac{b}{2a}) + c = 0 + \frac{b^2}{4a}$$

Factor

$$a(x + \frac{b}{2a})^2 + c = 0 + \frac{b^2}{4a}$$

Isolate the binomial

$$a(x + \frac{b}{2a})^2 = \frac{b^2}{4a} - c$$

$$a(x + \frac{b}{2a})^2 = \frac{b^2}{4a} - \frac{4a^2c}{4a}$$

$$\frac{a}{(a)}(x + \frac{b}{2a})^2 = \frac{b^2 - 4a^2c}{4a\,(a)}$$

Square root both sides

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4a^2c}}{2a}$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4a^2c}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Comparison: Example $\quad x^2 + 5x - 12 = 0$

Completing the square:

$$x^2 + 5x - 12 = 0$$

$$x^2 + 5x + \frac{25}{4} = 12 + \frac{25}{4}$$

$$(x + \frac{5}{2})^2 = \frac{73}{4}$$

$$x + \frac{5}{2} = \pm\sqrt{\frac{73}{4}}$$

$$x = \frac{-5}{2} \pm \frac{\sqrt{73}}{2}$$

Quadratic Formula:

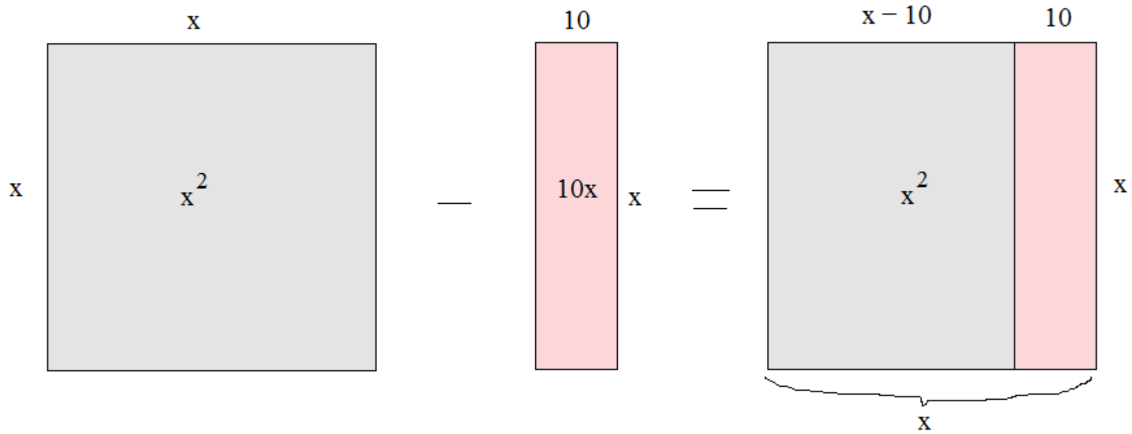
$a = 1$
$b = 5$
$c = -12$

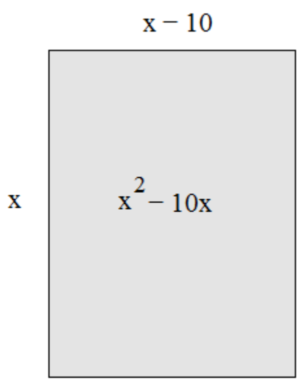
$$x = \frac{-(5) \pm \sqrt{(5)^2 - 4(1)(-12)}}{2(1)}$$

$$x = \frac{-5 \pm \sqrt{73}}{2}$$

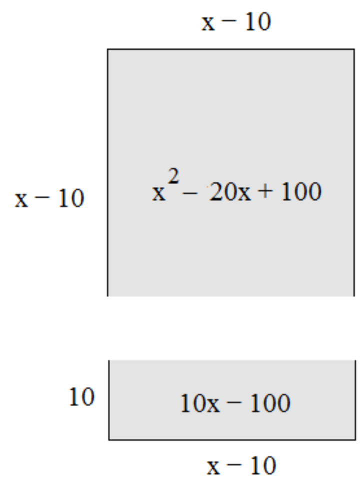
Completing the square (geometrically)

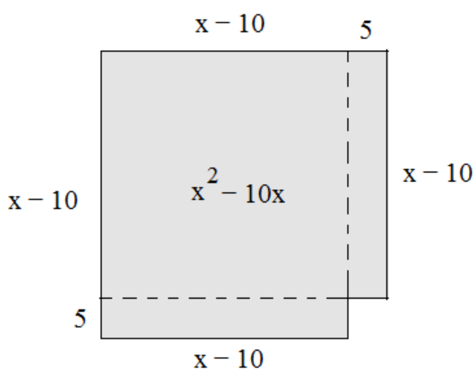
$x^2 - 10x$



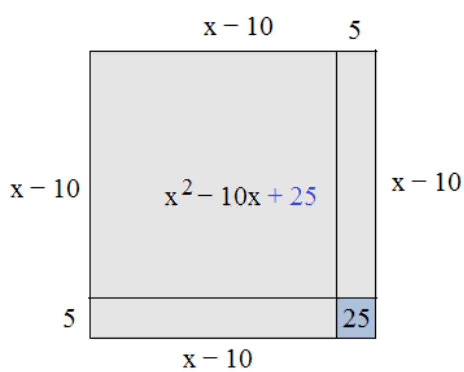


To create a square, the sides must be equal. So, we must subtract 10 from the vertical sides and redistribute...





We need a small piece to complete the square:



Completing the Square and Quadratic Formula Quiz

I. Factor by Completing the Square

a) $x^2 + 4x + 8$

b) $x^2 + 3x - 4$

c) $-x^2 + 4x + 9$

d) $4x^2 - 8x + 17$

e) $\frac{x^2}{3} + 2x + 10$

f) $5x^2 + 3x + 1$

II. Solve by Completing the Square

a) $x^2 + 4x = 7$

b) $-2x^2 + 9x + 3 = 0$

c) $x^2 + 6x = -11$

III. Solve using the Quadratic Formula

a) $x^2 + 7x - 3 = 0$

b) $2x^2 + 8x = -4$

c) $-x^2 + 3x + 5 = 0$

Completing the Square and Quadratic Formula Quiz

SOLUTIONS

I. Factor by Completing the Square

a) $x^2 + 4x + 8$

$x^2 + 4x \quad + 8$ — separate

$x^2 + 4x + 4 \quad + 8 - 4$ — $\left(\frac{b}{2}\right)^2$

$(x + 2)(x + 2) \quad + 4$ — factor the trinomial

$\boxed{(x + 2)^2 + 4}$

b) $x^2 + 3x - 4$

$x^2 + 3x \quad - 4$

$x^2 + 3x + \frac{9}{4} - 4 - \frac{9}{4}$

$\boxed{\left(x + \frac{3}{2}\right)^2 - \frac{25}{4}}$

note: 3/2 is $\frac{b}{2}$

c) $-x^2 + 4x + 9$

$-1(x^2 - 4x - 9)$ — change 1st coefficient to 1

$-1(x^2 - 4x \quad - 9)$ — separate

$-1(x^2 - 4x + 4 - 9 - 4)$ — $\left(\frac{b}{2}\right)^2$

$-1((x - 2)^2 - 13)$ — factor the trinomial

$\boxed{13 - (x - 2)^2}$

d) $4x^2 - 8x + 17$

$4x^2 - 8x \quad + 17$ — separate

$4(x^2 - 2x \quad) + 17$ — "change a to 1"

$4(x^2 - 2x + 1) + 17 - 4$ — $\left(\frac{b}{2}\right)^2$

add 4 — subtract 4 — factor the trinomial

$\boxed{4(x - 1)^2 + 13}$

e) $\frac{x^2}{3} + 2x + 10$

$\frac{x^2}{3} + 2x + \quad 10$

$\frac{1}{3}(x^2 + 6x \quad) + 10$

$\frac{1}{3}(x^2 + 6x + 9) + 10 - \frac{9}{3}$

$\boxed{\frac{1}{3}(x + 3)^2 + 7}$

f) $5x^2 + 3x + 1$

$5x^2 + 3x \quad + 1$

$5(x^2 + \frac{3}{5}x \quad) + 1$ — $\left(\frac{3/5}{2}\right)^2 = \frac{9}{100}$

$5(x^2 + \frac{3}{5}x + \frac{9}{100}) + 1 - \frac{45}{100}$

$\boxed{5\left(x + \frac{3}{10}\right)^2 + \frac{11}{20}}$

$\frac{b}{2a}$

II. Solve by Completing the Square

a) $x^2 + 4x = 7$

$x^2 + 4x + 4 = 7 + 4$ — add $\left(\frac{b}{2}\right)^2$ to both sides

$(x + 2)(x + 2) = 11$

$\sqrt{(x + 2)^2} = 11$

$(x + 2) = \pm\sqrt{11}$

$\boxed{x = -2 \pm \sqrt{11}}$

b) $-2x^2 + 9x + 3 = 0$

$-2(x^2 + -\frac{9}{2}x) = -3$

$-2(x^2 - \frac{9}{2}x + \frac{81}{16}) = -3 + \frac{81}{16}(-2)$

$-2(x + \frac{9}{4})^2 = \frac{-105}{8}$

$(x + \frac{9}{4})^2 = \frac{105}{16}$

$x + \frac{9}{4} = \pm\frac{\sqrt{105}}{4}$

$\boxed{x = \frac{9}{4} \pm \frac{\sqrt{105}}{4}}$

c) $x^2 + 6x = -11$

$x^2 + 6x + 9 = -11 + 9$

$(x + 3)^2 = -2$

$x + 3 = \pm\sqrt{-2}$ — No REAL solutions

$\boxed{x = -3 \pm i\sqrt{2}}$

III. Solve using the Quadratic Formula

a) $x^2 + 7x - 3 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

a = 1
b = 7
c = -3

$x = \frac{-(7) \pm \sqrt{(7)^2 - 4(1)(-3)}}{2(1)}$

$\boxed{x = \frac{-7 \pm \sqrt{61}}{2}}$

b) $2x^2 + 8x = -4$

$2x^2 + 8x + 4 = 0$

a = 2
b = 8
c = 4

$x = \frac{-(8) \pm \sqrt{(8)^2 - 4(2)(4)}}{2(2)}$

$x = \frac{-8 \pm \sqrt{32}}{4}$

$\boxed{x = -2 \pm \sqrt{2}}$

c) $-x^2 + 3x + 5 = 0$

a = -1
b = 3
c = 5

$x = \frac{-(3) \pm \sqrt{(3)^2 - 4(-1)(5)}}{2(-1)}$

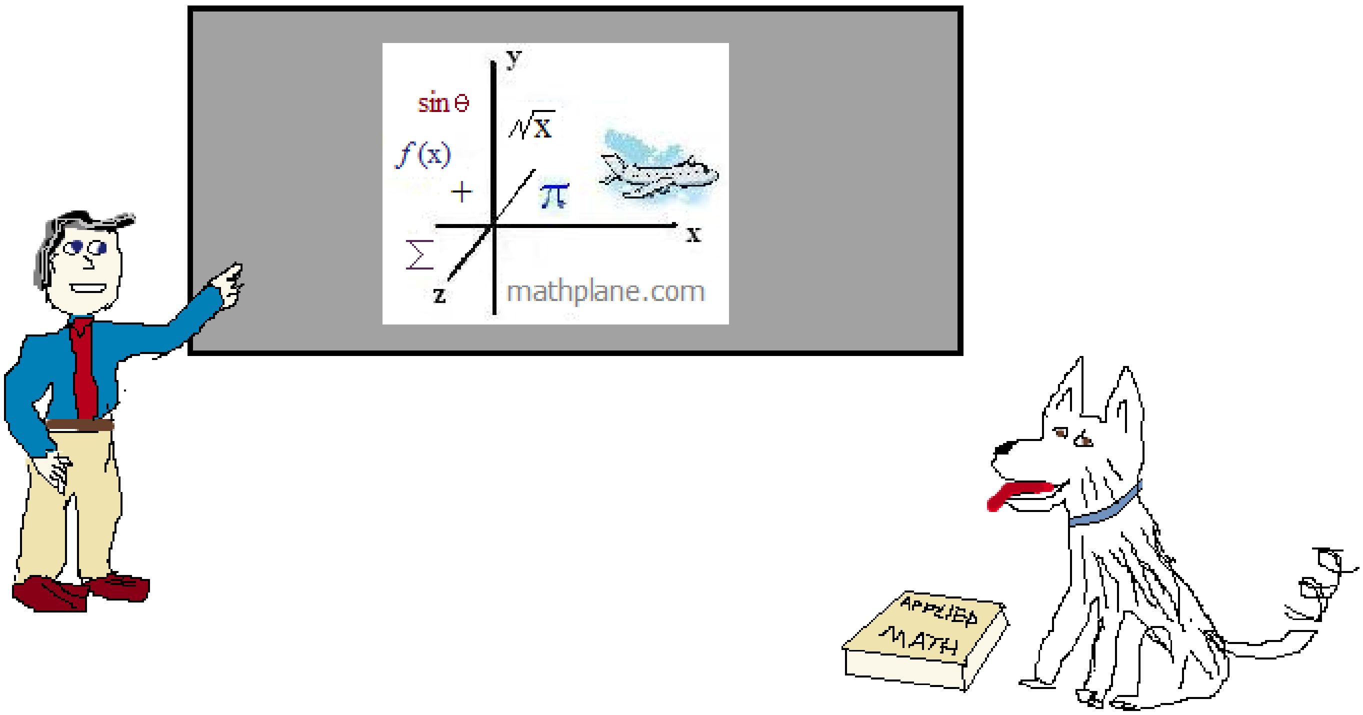
$x = \frac{-3 \pm \sqrt{29}}{-2}$

$\boxed{x = \frac{3 \mp \sqrt{29}}{2}}$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers



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